

# Histories and Quantum Speed Limit of Neutrino Oscillations under External Conditions

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## Abstract

The aim of this dissertation is to investigate various properties of Neutrino Oscillations when subjected to external conditions, and touch on the impact of external conditions on properties of general finite dimensional systems. This investigation shows that to shed further light on quantum measurement, and dynamical properties of such systems, they need to be studied as *open* systems. This is actualized as a compilation of peer-reviewed articles that cover properties of neutrino oscillations like Flavor Transition Probability, Stead-State autocorrelation function, Geometric Speed Limit, and consistency of family of histories of neutrino oscillations using the unorthodox Consistent Histories Approach. The external conditions considered are Continuous non-selective measurement, presence of normal matter as environment, CP-violating phase, and presence of Decoherence. Presence of Normal Matter increases the probability of change of flavor from the initial electron flavor. CP violating phase damps only the troughs of the oscillations of flavor transition probability, while Decoherence damps the crests instead. Continuous, non-selective measurement significantly increases the electron to muon flavor transition, and damps the oscillations for electron to electron or electron to tau transitions. The investigation shows that Geometric Quantum Speed Limit can serve as a tool that quantifies CP-Violation barring time windows where it is independent of CP violating phase. The dissertation highlights the magnitude of inaccessibility of neutrino mass measurement by showing that it is *accessible*, in the sense of neutrino being in a pure mass state being a *consistent quantum event*, only for particular values of the dynamics of neutrino oscillation. It is hardly accessible, but not *fundamentally impossible*.

# Contents

|          |                                                                            |           |
|----------|----------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction</b>                                                        | <b>6</b>  |
| <b>2</b> | <b>Neutrino Oscillation</b>                                                | <b>8</b>  |
| 2.1      | Introduction . . . . .                                                     | 8         |
| 2.2      | Neutrino Oscillation . . . . .                                             | 8         |
| 2.3      | Qutrit Approximation . . . . .                                             | 9         |
| 2.4      | Qubit Approximation . . . . .                                              | 11        |
| <b>3</b> | <b>Continuous Non-Selective Measurement</b>                                | <b>13</b> |
| 3.1      | Introduction . . . . .                                                     | 13        |
| 3.2      | Quantum Master Equation for Continuous Non-Selective Measurement . . . . . | 14        |
| <b>4</b> | <b>Geometric Speed Limit</b>                                               | <b>16</b> |
| 4.1      | Introduction . . . . .                                                     | 16        |
| 4.2      | Formulation of Quantum Speed Limit Time . . . . .                          | 16        |
| 4.3      | Formulation of Geometric Speed Limit . . . . .                             | 17        |
| <b>5</b> | <b>Consistent Histories Approach</b>                                       | <b>20</b> |
| 5.1      | Introduction . . . . .                                                     | 20        |
| 5.2      | Quantum Sample Space . . . . .                                             | 20        |
| 5.3      | Quantum Histories . . . . .                                                | 23        |
| 5.4      | Assigning Probability to Histories . . . . .                               | 24        |

|          |                                                                                                              |           |
|----------|--------------------------------------------------------------------------------------------------------------|-----------|
| 5.5      | Consistency Condition . . . . .                                                                              | 25        |
| <b>6</b> | <b>(Article A) Neutrino Oscillations in the Presence of Matter and Continuous Non-Selective Measurement</b>  | <b>26</b> |
| 6.1      | Introduction . . . . .                                                                                       | 26        |
| 6.2      | Model . . . . .                                                                                              | 27        |
| 6.3      | Results . . . . .                                                                                            | 28        |
| 6.4      | Conclusion . . . . .                                                                                         | 30        |
|          | The Article . . . . .                                                                                        | 31        |
| <b>7</b> | <b>(Article B) Geometric speed limit of neutrino oscillation</b>                                             | <b>42</b> |
| 7.1      | Introduction . . . . .                                                                                       | 42        |
| 7.2      | Model . . . . .                                                                                              | 43        |
| 7.3      | Results . . . . .                                                                                            | 45        |
| 7.4      | Conclusion . . . . .                                                                                         | 47        |
|          | The Article . . . . .                                                                                        | 49        |
| <b>8</b> | <b>(Article C) Histories of Neutrino Oscillation of Consistency Induced by the Presence of Normal Matter</b> | <b>65</b> |
| 8.1      | Introduction . . . . .                                                                                       | 65        |
| 8.2      | Model . . . . .                                                                                              | 66        |
| 8.3      | Results . . . . .                                                                                            | 68        |
| 8.4      | Conclusion . . . . .                                                                                         | 72        |
|          | The Article . . . . .                                                                                        | 74        |

|           |                                                                                                                                                        |            |
|-----------|--------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| <b>9</b>  | <b>(Article D) Consistency of Histories of Neutrino Oscillation<br/>in the Presence of Normal Matter and Continuous Non-<br/>Selective Measurement</b> | <b>85</b>  |
| 9.1       | Introduction . . . . .                                                                                                                                 | 85         |
| 9.2       | Model . . . . .                                                                                                                                        | 85         |
| 9.3       | Results . . . . .                                                                                                                                      | 87         |
| 9.4       | Conclusion . . . . .                                                                                                                                   | 89         |
|           | The Article . . . . .                                                                                                                                  | 91         |
| <b>10</b> | <b>Conclusions</b>                                                                                                                                     | <b>105</b> |
| <b>11</b> | <b>Author Contributions</b>                                                                                                                            | <b>111</b> |

# 1 Introduction

This doctoral dissertation, titled "Histories and Quantum Speed Limit of Neutrino Oscillations under External Conditions", is a compilation of a series of articles published in peer-reviewed journals. There are four articles that form the basis of this dissertation, and they are as follows:

- (A) Khan, F.; Dajka, J. Neutrino Oscillations in the Presence of Matter and Continuous Non-Selective Measurement. *Symmetry* 2020, 12, 1296.  
<https://doi.org/10.3390/sym12081296>
- (B) Khan, F., Dajka, J. Geometric speed limit of neutrino oscillation. *Quantum Inf Process* 20, 193 (2021).  
<https://doi.org/10.1007/s11128-021-03128-7>
- (C) Khan, F.; Dajka, J. Histories of Neutrino Oscillation of Consistency Induced by the Presence of Normal Matter. *Universe* 2022, 8, 106.  
<https://doi.org/10.3390/universe8020106>
- (D) Khan, F.A.; Dajka, J. Consistency of Histories of Neutrino Oscillation in the Presence of Normal Matter and Continuous Non-Selective Measurement. *Universe* 2025, 11, 57.  
<https://doi.org/10.3390/universe11020057>

Collectively, the articles investigate the impact of external conditions on the properties of Neutrino Oscillations, with each article considering a set of different properties of the system and external conditions applied to that system. For cohesion of the articles and this dissertation, the system was taken to be a single neutrino starting in an *initial state* of that of an electron neutrino flavor state.

The first article studies two properties of the system, the probability of transition of an electron neutrino, and a Steady-State autocorrelation function. These properties are computed with and without the presence of a Continuous Non-Selective Measurement under the absence of normal matter, presence of normal matter, with a vanishing CP-violating phase, and with a hypothetical non-vanishing CP-violating phase.

The second article discusses the transition probability again, and Geometric Speed Limit for a neutrino in the initial electron flavor state. These are sub-

jected to the absence and presence of normal matter, absence and presence of Decoherence, and a vanishing and non-vanishing CP-violating phase.

The third article designs a family of histories of neutrino oscillations between flavor states and pure mass states, and pursues the consistency of such a family of histories under what is termed the "Consistent Histories Approach" (also called Decoherent Histories) and explores whether the absence and/or presence of normal matter, and a vanishing and/or non-vanishing CP-violating phase impacts the consistency of this family of histories.

The fourth article combines the first and third articles. It utilises the family of histories designed in the third article and applies the model of Continuous Non-Selective Measurement from the first article. It studies the impact of measurement on the consistency of this family of histories.

While this dissertation presents a particular system, it qualitatively touches the general topics of impact of external conditions on properties of finite-dimensional systems and the requirement of such systems to be realistically studied as open systems to shed light on these dynamical properties, and quantum measurement as a process itself.

The articles in the dissertation also emphasize the aspects of neutrino oscillations that make it an objective for interdisciplinary research. Article C in particular also speaks about the current inaccessibility of measurement of mass of neutrinos, and the question of the nature of neutrino as to whether it is a Dirac or Majorana type particle.

The dissertation cites references from the articles themselves and uses the notation of "[B2]" to refer to the reference in the second place in the bibliography of Article B.

**Keywords:** Neutrino Oscillation, Open Systems, Quantum Measurement

## 2 Neutrino Oscillation

### 2.1 Introduction

Time-evolving neutrinos [A1,C1] attract attention and stimulate research from various academic perspectives [A2,A3], and have even entered the realm of science fiction [B25,C15]. Despite enormous progress in experimental techniques, even the latest and most sophisticated experiments [C3] do not resolve all doubts about the fundamental properties of these mysterious and fascinating particles.

The dynamic properties of neutrino oscillations [A4,A5,C4–C6], initially studied in the context of particle physics [A1,C1], have already been related to quantum information processing [A5,A13,A14,A15,A16,C5,C7–C13]. Some of these studies range from recent proposals of information transfer protocols utilizing neutrinos [C14] as a resource, to the human dream of interstellar communication [B25,C15], and to research related to decoherence [A4,C4,A7–12].

Neutrinos hardly interact with anything. Thus, time-evolving neutrinos [A1,C1,C2] and their oscillations are an example of quantum systems that are particularly difficult to measure and can feature in such research as well.

In principle, Quantum information processing models [B46] such as circuit model [B46], topological models, Zidan’s model of quantum computing proposed in [B56,B57] or quantum communication protocols [B36,B45,B48] can be realized using neutrinos [B52]. Such models, along with models using gravitational waves [B1] as resource, still remain in a domain of science fiction [B25,C15] rather than reality.

### 2.2 Neutrino Oscillation

As succinctly put in [A1,C1], no experiments that have been performed so far have detected *conclusive* deviations from the Standard Model, except neutrino oscillation experiments, which have shown that neutrinos are massive and mixed. However, no deviations from the standard neutrino interactions have been found in the experimental data. Neutrino interactions are very weak. They are known to interact through charged current or neutral

current weak interactions [A1,C1].

Neutrino mass has also been a topic of important discussion. Although we have yet to know the neutrino masses, two small values of squared-mass differences have been experimentally determined [A18,B18,C38]. Neutrinos are produced in a weak interaction alongside a charged lepton, electron, muon or tau. This leads to the qutrit formulation for neutrino oscillations, sometimes simplified to a qubit approximation to study properties of neutrinos qualitatively.

There is a non-correspondence between the flavor states and the mass states of neutrinos. Due to this, a neutrino of a given flavor can be measured in a different mass state after propagating for some time and distance [A1,C1]. This phenomena is what is referred to as Neutrino Oscillation.

### 2.3 Qutrit Approximation

Qutrit approximation for neutrinos employs a 3-level system. A neutrino can have three flavor states  $\{v_e, v_\mu, v_\tau\}$  that have a non-correspondence with the three mass states  $\{v_1, v_2, v_3\}$ . The flavor and mass states are connected through Pontecorvo-Maki-Nakagawa-Sakata unitary lepton mixing matrix  $U_{PMNS}$ .

$$\begin{pmatrix} v_e \\ v_\mu \\ v_\tau \end{pmatrix} = U_{PMNS} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \cdot P$$

The matrix  $P$  is given as:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$

$c_{ij} = \cos(\theta_{ij})$  and  $s_{ij} = \sin(\theta_{ij})$ . There are three mixing angles  $\theta_{12}$ ,  $\theta_{23}$ , and  $\theta_{13}$ , one Dirac type CP-violating phase  $\delta$ , and two Majorana type CP-violating phases  $\alpha$  and  $\beta$ .

Whether neutrino is a Dirac type, or Majorana type particle is still an open question. The results obtained at the current level of experimental

sensitivity have been compatible with CP-Symmetry ( $\delta = 0$ ), however the most recent experiments [B54] indicate that  $\delta$  can be non-vanishing, which the articles refer to as a hypothetical CP violation in the articles.

Articles A and B of this dissertation have considered Dirac type neutrinos. Thus, the Majorana phases do not feature in the mixing matrix, and it takes the form below:

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

When it comes to the dynamics of neutrino oscillation, the total Hamiltonian  $H_F$  is given as a sum of the Kinetic and Potential parts [A1,A19-A21,C1]:

$$H_F = H_{kin} + H_{pot}.$$

Accounting for the ultra-relativistic limit for the neutrinos, the Eigenenergies are given as  $E_k = E + \frac{m_k^2}{2E}$ , and  $p \sim E$ , where  $E$  is the energy of the massless neutrino. Thus, the kinetic part of the Hamiltonian in the flavor basis reads as

$$H_{kin} = \frac{1}{2E} U_{PMNS} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} U_{PMNS}^\dagger,$$

where  $\Delta m_{ij}^2 = m_i^2 - m_j^2$  is the difference between the masses.

For the potential part, the main contribution originates from neutrino's interaction (via coherent forward elastic scattering) with the matter's electrons and the weak charge of the nuclei generating charged-current potential  $V_C$ . As per the standard model, the neutral current interaction does not enter effectively into the Hamiltonian [A1,C1]. For all three articles, the propagation was considered to be in normal matter only, due to which charged current interaction term for muon is not included. The potential part reads as

$$\begin{pmatrix} V_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $V_C$  is related to the electron's density in the matter and is given as  $V_C = \sqrt{2}G_F n_e$ .  $G_F$  is the Fermi Coupling constant, and  $n_e$  is the electron number density.

## 2.4 Qubit Approximation

Qubit approximation for neutrinos employs a 2-level system. A neutrino can have two flavor states  $\{v_e, v_\mu\}$  that have a non-correspondence with the two mass states  $\{v_1, v_2\}$ . Such an approximation is practically useful as it allows for simpler mixing parameters and often has analytical solutions possible, and as many experiments are not sensitive to the three-neutrino mixing, the analysis can be carried out for those experiments effectively with qubit approximation [A1,C1].

Article C and D use the Qubit approximation for neutrino oscillation. They also employ a CP-violating Majorana phase in the dynamics, as it is non-zero under  $U(1)$  global transformation, unlike the Dirac case, where this transformation renders it zero [A1,C1].

The flavor and mass states are still connected through Pontecorvo-Maki-Nakagawa-Sakata unitary lepton mixing matrix  $U_{PMNS}$ , and the two-level approximation is achieved by substituting  $\theta_{13} = \theta_{23} = 0$ .

$$\begin{pmatrix} v_e \\ v_\mu \end{pmatrix} = U_{PMNS} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

$$U_{PMNS} = \begin{pmatrix} \cos(\theta_{12}) & \sin(\theta_{12}) \\ -\sin(\theta_{12}) & \cos(\theta_{12}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

$\theta_{12}$  is the mixing angle, and  $\phi$  is the CP-violating Majorana phase.

Thus, the flavor states are given as

$$\begin{aligned} |v_e\rangle &= \cos\theta_{12} |1\rangle + e^{i\phi} \sin\theta_{12} |2\rangle \\ |v_\mu\rangle &= -\sin\theta_{12} |1\rangle + e^{i\phi} \cos\theta_{12} |2\rangle. \end{aligned}$$

The vacuum Hamiltonian for a two-level neutrino is given as

$$H_0 = \begin{pmatrix} E - \frac{\Delta m_{21}^2}{4E} & 0 \\ 0 & E + \frac{\Delta m_{21}^2}{4E} \end{pmatrix}$$

The effective Hamiltonian for neutrino propagating in normal matter, called Wolfenstein effective Hamiltonian [A1,A4,C1,C4], is given as:

$$H = H_0 + \frac{V_0}{2} \begin{pmatrix} 1 + \cos 2\theta_{12} & e^{-i\phi} \sin 2\theta_{12} \\ e^{i\phi} \sin 2\theta_{12} & 1 - \cos 2\theta_{12} \end{pmatrix},$$

where  $V_0$  is the effective interaction potential due to oscillation of the neutrino in the presence of normal matter. As per the standard model, the neutral current interaction does not enter effectively into the Hamiltonian, charged current interaction term for muon is not included as only normal matter is considered. Thus, the potential comes from the coherent forward scattering on electrons via charged current interactions [A1,C1].

$V_0 = \sqrt{2}\cos^2\theta_{13}G_F n_e$ .  $G_F$  is the Fermi Coupling constant,  $n_e$  is the electron number density, and  $0.953 \leq \theta_{13} \leq 1$  is one of the oscillation parameters with  $3\sigma$  bound [A1,C1].

## 3 Continuous Non-Selective Measurement

### 3.1 Introduction

Quantum Measurements are understood to cause dynamical changes that their classical counterparts are not subject to. This places a limit on the information that can be obtained about the state of the system [A23]. The ideal von Neumann Scheme, where the system being measured and the apparatus carrying out the measurement *interact* to give what we call a *measurement*, even when taken to be ideal (Quantum non-demolition measurement), is fundamentally different from the classical idea of measurement.

One can hardly perform a direct measurement on a Quantum System. In real systems, one sort of allows the system that is to be measured to interact with a *probe*, which is another system in itself, but one that has properties accessible to read as classical apparatus. Such measurement are fundamentally indirect, and can be summarised as a two part process:

- (i) Observable  $A$  of a quantum system to be measured couples to a probe *position*  $Q$  resulting in an interaction between system and probe.
- (ii) Probe *momentum*  $P$  canonically conjugated to  $Q$  is projectively measured

The first part can be referred to as the act of making the *measurement*, while the second can be thought of as *reading* the measurement made. That's why real experiments that are associated with measuring physical quantities almost never can be sufficiently described in terms of von-Neumann scheme [A17,A22,A23].

Going with the analogy of measurement being a two part process, a non-selective measurement is a measurement with the result ignored or not "read". A continuous measurement is where information is continually extracted from a system [A17]. A general way to make a *continuous* measurement is to divide time into a sequence of intervals of length  $\Delta t$ , and consider a weak measurement in each interval. The strength of each measurement is made proportional to the time interval, and then take the limit in which the time intervals become infinitesimally short.

### 3.2 Quantum Master Equation for Continuous Non-Selective Measurement

Going by the two part analogy of measurement, the Hamiltonian for the first step is system-probe total Hamiltonian:

$$H_{sys+probe} = H_{sys} + H_{probe} + g(t)H_{int}$$

where observable  $A$  of a quantum system couples with probe position  $Q$  to give the interaction term  $H_{int} = A \otimes Q$ .  $g(t)$  is the strength of the coupling.

If the measurement is taken to be non-selective, that is a measurement with result ignored, and the time interval is divided into intervals of length  $q$ , the dynamics of the system, after being reduced with respect to the probe, are no longer unitary, however they form a quantum complete positive map [A24]. That is a map that maps quantum states (positive operators) onto quantum states.

Without complete positivity, the dynamics lead to a physically inconsistent time-evolution [B12], and is vital for constructing reduced non-unitary dynamics, so that after reducing out the probe from the composite  $H_{sys} \otimes H_{probe}$  system, one still gets a map  $\Lambda_t : H_{sys} \rightarrow H_{sys}$ .

Particularly, for a set of measurement operators  $E_x$ , where  $\sum_x E_x^\dagger E_x = \mathbb{I}$ , non-selective measurement of a system in state  $\rho$  results in a transformation  $\rho \rightarrow E_x \rho E_x^\dagger$ , which is an example of a complete positive map [A22].

Thus, such a measurement on each time interval of length  $q$  results in the evolution of the state as

$$\rho(t+q) = \int dp E_p \rho(t) E_p^\dagger,$$

with  $E_p = e^{iqH_{sys}} \langle p | e^{-iqH_{int}} | \phi_q \rangle \cdot | p \rangle \in H_{probe}$  is an eigenstate of  $P$ , the probe momentum, and  $| \phi_q \rangle \in H_{probe}$  is state of the probe before the measurement.

For a finite time duration of the measurement, if the above equation for  $\rho(t+q)$  is expanded to the second order in  $q$ , in order to get a stochastic equation in linear form, one arrives to a quantum master equation below:

$$\frac{d}{dt}\rho(t) = -i[H_{sys}, \rho(t)] - \frac{\sigma}{2}[A, [A, \rho(t)]].$$

The first term represents the information gain due to the measurement process, and the second represents the disturbance to, or the back-action on, the state of the system due to the measurement [A24].

The parameter  $\sigma$  describes the back-action of the measurement. Taking the coupling strength to be proportional to the length of time interval  $q \sim \int_0^t g(s)ds$ ,  $\sigma$  is formulated as

$$\sigma^2 = \lim_{q \rightarrow 0} q \langle \phi_q | Q^2 | \phi_q \rangle$$

where the limit describes passage from a step-like to a truly continuous measurement.

The more accurately the observable  $A$  is measured, the more information is gained about it, the larger back-action is observed from the measurement, resulting in larger value for  $\sigma$  [A25]. Thus, if one defines  $t_z = \frac{1}{\sigma^2}$ , the limit  $t_z \rightarrow 0$  reaches the Quantum Zeno effect [A25,A26], where the time evolution of the system suppresses due to frequent measurement of the system. A finite value for  $\sigma$  can then be understood as *non-ideal Zeno effect*.

## 4 Geometric Speed Limit

### 4.1 Introduction

Heisenberg Uncertainty Principle embeds the inherently probabilistic interpretation of Quantum Theory. Applying it to position and momentum, the modern interpretation firmly understands it to state that the position and the momentum of a quantum particle cannot be measured simultaneously with infinite precision.

This, as reviewed in [B28] can be seen as an interpretation of the Heisenberg Uncertainty Principle about simultaneous events, however, the interpretation of an uncertainty in time,  $\Delta E \Delta t \gtrsim \hbar$ , is not obvious. This opened the idea for Quantum Limits of Evolution and led to the formulation of Quantum Speed Limits (QSL).

Mandelstam and Tamm [B42] realized that the Energy-Time Uncertainty Relation is about the time scale of the quantum dynamics, that is that  $\Delta t$  can be seen as the time a quantum system needs to evolve from an initial to a final state, formalized as the *Quantum Speed Limit Time*,  $T_{QSL}$ . This started the chain of formulating and studying quantum limit of evolution time from various perspectives, like those in [B7,B21,B27,B30,B34,B55]: quantum control, quantum information, or metrology.

*Geometric QSL*, which was proposed in [B29] allows to identify how a rate of information, given by a change of statistical distance, is related to dynamical properties of quantum dynamical systems, governed by a very general class of equations of motion.

### 4.2 Formulation of Quantum Speed Limit Time

For Quantum Systems evolving under Schrödinger dynamics, the evolution of any observable  $A$  is described by the Liouville-von-Neumann equation,

$$\frac{\partial A}{\partial t} = \frac{i}{\hbar}[H, A],$$

where  $H$  is the Hamiltonian of the system.

For any two operators  $A$  and  $B$ , through Cauchy-Schwarz inequality, one gets

$$\Delta A \Delta B \geq \frac{1}{2} | \langle [A, B] \rangle |$$

where for  $O = A, B$ ,  $\Delta O \equiv \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$ , and  $\langle O \rangle \equiv \langle \psi | O | \psi \rangle$  [B28].

Applying the general uncertainty relation to the Liouville-von-Neumann equation, one gets

$$\Delta H \Delta A \geq \frac{\hbar}{2} \left| \left\langle \frac{\partial A}{\partial t} \right\rangle \right|.$$

To further simplify, the observable is chosen to be a projector onto the initial state  $|\psi(0)\rangle$ , and  $A = |\psi(0)\rangle\langle\psi(0)|$ . This leads to  $\langle A \rangle_0 = 1$ ,  $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} = \sqrt{\langle A \rangle - \langle A \rangle^2}$ , and then integrating the uncertainty relation for the Liouville-von-Neumann equation above, the following equation is obtained:

$$\frac{1}{\hbar} \Delta H t \geq \frac{\pi}{2} - \arcsin \sqrt{\langle A \rangle_t}.$$

Then, if one considers a process where the final state is orthogonal to the initial state,  $\langle \psi(0) | \psi(\tau) \rangle = 0$ , one arrives at the minimal time for a quantum system to evolve between two orthogonal states [B28], and it is given as

$$\tau \geq \tau_{QSL} = \frac{\pi}{2} \frac{\hbar}{H}.$$

The above is not an uncertainty relation between duration of a measurement and the energy transferred to the observed system. Quantum Speed Limit time  $\tau_{QSL}$  actually sets an essential time scale of any quantum evolution [B28].

### 4.3 Formulation of Geometric Speed Limit

Formulation of Geometric QSL relies on some concepts from Quantum Information Theory. The first of them is Fidelity, or the Fidelity Function  $F$ .

*Fidelity* represents the probability of one state passing the test to identify as the other. It can also be thought of as a measure of closeness of two quantum

states. If both states are pure, Fidelity equals the transition probability between the states in question [B13].

*Root Fidelity*, which is the square root of the Fidelity function is given as

$$\sqrt{F(\rho_1, \rho_2)} = \text{Tr}|\sqrt{\rho_2}\sqrt{\rho_1}|.$$

*Bures Angle*,  $l(t)$ , is a measure of the statistical distance between quantum states. It is a generalization to mixed states of the angle in Hilbert space between two state vectors [B46], and it follows from the concept of Root Fidelity:

$$l(t) = \arccos \sqrt{F(\rho(0), \rho(t))}.$$

$\rho(0)$  represents the density matrix for the initial state, while  $\rho(t)$  is the density matrix for the state after time  $t$ . Fidelity gives the closeness between the initial and *final* state, and Bures Angle geometrically represents the *distance* between these two states.

To formulate Geometric QSL, one needs to consider the evolution of the density operator of the system, and can do so by taking the time derivative of the Bures Angle [B29].

$$\frac{d}{dt}l(t) \leq \left| \frac{d}{dt}l(t) \right| = \frac{1}{\sqrt{1 - F(\rho(0), \rho(t))}} \frac{|F(\rho(0), \dot{\rho}(t))|}{2\sqrt{F(\rho(0), \rho(t))}}$$

By definition of  $l(t)$ , one gets that  $\cos l(t) = \sqrt{F(\rho(0), \rho(t))}$ , which further leads to  $\sin l(t) = \sqrt{1 - F(\rho(0), \rho(t))}$ , the time derivative further simplifies to:

$$2\cos l(t)\sin l(t) \dot{l}(t) \leq |F(\rho(0), \dot{\rho}(t))|.$$

The right hand side can be expanded for the simpler case of Hamiltonian for the system and density operator evolving according to von Neumann equation. As article B considers an evolution governed by a Markovian master equation [B4,B20], the right hand side is expanded using

$$\dot{\rho}_t = \mathbb{L}[\rho_t],$$

where  $\mathbb{L}$  is an arbitrary non-unitary positive generator. The positivity requirement comes from the need to eliminate any physical inconsistencies arriving from lack of complete positivity [B10,B12].

Substituting the evolution of the density operator to the time derivative of Bures Angle reads as

$$2\cos l(t)\sin l(t) \dot{l}(t) \leq \left| F(\rho(0), \mathbb{L}[\rho(t)]) \right| = \left| \text{Tr}[\mathbb{L}\rho(t)\rho(0)] \right|.$$

Singular values of an operator  $A$  are defined as the eigenvalues of  $\sqrt{A^\dagger A}$ . By von Neumann trace inequality,  $\text{Tr}[\mathbb{L}\rho(t)\rho(0)] \leq \sum_i \sigma_i p_i$ , where  $\sigma_i$  are the singular values of  $\mathbb{L}\rho(t)$ , and  $p_i = \delta_{i,1}$ . Thus,  $\sum_i \sigma_i p_i = \sigma_1$ , representing the largest singular value, and given as operator norm  $\|\mathbb{L}\rho(t)\| = \sigma_{\max}$ .

Therefore, the time derivative for the Bures Angle can be further written down as

$$2\cos l(t)\sin l(t) \dot{l}(t) \leq \|\mathbb{L}\rho(t)\|.$$

Integrating with respect to time, one reaches the Geometric QSL,  $\nu_{QSL}$ , written as

$$\tau \geq \nu_{QSL} = \frac{\|\mathbb{L}\rho(t)\|}{2\cos l(t)\sin l(t)}.$$

Thus, the Geometric QSL is founded on Bures Angle, and by definition allows to identify how a rate of information, given by a change of statistical distance, is related to dynamical properties of a quantum dynamical system [B29].

As Geometric QSL comes from the definition of rate of change of Bures Angle, the quantum speed limit time becomes the inverse of the average *speed* [B28], and the two become related as:

$$\frac{1}{\tau_{QSL}} = \frac{1}{t} \int_0^t \nu_{QSL}(t') dt'.$$

## 5 Consistent Histories Approach

### 5.1 Introduction

*Consistent Histories Approach* is an alternative to the orthodox *Copenhagen Interpretation* of Quantum Theory. In this interpretation, Quantum Systems are still inherently probabilistic, however it does not rely on the notion of measurement itself to assign probabilities to outcomes of measurements. In doing so, it moves away from the standard interpretation of the act of measurement *collapsing* the wave function of the system onto one of the eigenstates of the observable being measured.

A limitation of the standard interpretation is that one cannot talk about a quantum system between measurements at all. Quantum state embodies all the knowledge about the system, however one cannot conclusively say anything about the system, until the act of measurement collapses it into one of the eigenstates of the observable, and the celebrated *Born Rule* describes the probability of each possible outcome.

Consistent Histories approach, instead of using measurement, designs a probabilistic sample space instead, and attempts to describe the quantum system by treating the dynamics as stochastic. Measurement is like any other process then, and plays no role in the interpretation of the theory.

The real predictive power of Consistent Histories approach comes to the forefront when the quantum system in question is measurement resistive, or the measurement is not accessible. Thus, it becomes really useful to study neutrinos and therefore features in articles C and D.

The reliance on measurement of standard quantum interpretation gives rise to a lot of paradoxes, and the logical use of frameworks that shall be discussed in this chapter to resolve these paradoxes.

### 5.2 Quantum Sample Space

The foundation of Consistent Histories Approach rests on the formulation of what is referred to as a *Quantum Sample Space*. Classical probability theory is based on the concept of sample space of mutually exclusive possibilities,

such that only one of these occurs in any given situation [C21].

The elements of a classical sample space are called elements or *events*. For each sample space  $S$ , there is its associated event algebra  $B$ , which consists of subsets of elements of the sample space. An event that corresponds with one element of the sample space is called an *elementary* event, while an event that corresponds with more than one elements of the sample space is a *compound* event. The subsets of  $S$  form a Boolean Algebra as well, that is that the complement of subset  $\mathcal{E}$  of  $S$  is the set of elements of  $S$  not in  $\mathcal{E}$ , the intersection of two subsets of  $S$ ,  $\mathcal{E} \cap \mathcal{F}$ , is the set of elements they have in common, and their union  $\mathcal{E} \cup \mathcal{F}$  is the set of elements belonging to  $\mathcal{E}, \mathcal{F}$ , or both.

Thus, any probabilistic theory has 4 elements, which are:

- Sample Space
- Event Algebra
- Boolean Algebra
- Probability Distribution

Having a sample space, the indicator function  $E$  for an event  $\mathcal{E}$  on the sample space is given as

$$E(s) = \begin{cases} 1 & \text{if } s \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

These indicators also form a Boolean Algebra [B21].

For each element  $r$  of the sample space, there is a special indicator function which is 0 except at point  $r$ ,

$$P_r(s) = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } s \neq r \end{cases}$$

These special indicators are also elementary indicators as

$$P_r P_s = \delta_{rs} P_s.$$

That the product of two distinct elementary indicators of a sample space vanishes shows that the elements associated with the indicators are mutually exclusive [C21].

The indicator  $R$  for any compound event  $\mathcal{R}$  can be written as the sum of elementary indicators

$$R = \sum_{s \in \mathcal{S}} \pi_s P_s,$$

where  $\pi_s = 1$  if  $s \in \mathcal{R}$ , and 0 otherwise.

Lastly, the indicator  $I$  is the identity indicator that takes the value 1 everywhere in  $\mathcal{S}$  is then written as

$$I = \sum_{s \in \mathcal{S}} P_s$$

as  $\pi_s = 1$  for every  $s$ .

This completes the objects needed to construct a Quantum Sample Space working backwards. The Hermitian Projection Operator called the projector plays a vital role in Quantum Mechanics. A projector  $P$  satisfies two conditions,  $P^2 = P$ , and  $P^\dagger = P$ .

Each projector  $P$  has its own subspace  $\mathcal{P} \in \mathcal{H}$ , which consists of all state vectors that are invariant under this transformation, that is all  $|\psi\rangle$  such that  $P|\psi\rangle = |\psi\rangle$ . Projective decomposition of identity (PDI) is an orthogonal collection of projectors  $\{P_j\}$ , where  $P_j P_k = \delta_{jk} P_k$ , which sum to the identity

$$I = \sum_j P_j.$$

The projectors of quantum theory also meet the requirements of being special indicator functions. Therefore, an orthogonal collection of projectors  $P_j$  such that  $P_j P_k = \delta_{jk} P_k$ , and are elementary indicators. The Identity,  $I$ , of the Hilbert Space,  $\mathcal{H}$ , which is 1 everywhere, is thus:

$$I = \sum_j P_j.$$

Hence, the requirement of the sample space, which is being a set of mutually exclusive events, is met by the orthogonal decomposition of the identity (PDI), which makes up the Quantum Sample Space [C21].

Quantum Event Algebra for the sample space consists of all projectors of the form  $R = \sum_j \pi_j P_j$  where  $\pi_j = 1$  if  $j \in r$  and 0 otherwise. As the projectors forming the sample space are elementary indicators, these projectors from the event algebra form a Boolean algebra [C21].

### 5.3 Quantum Histories

Having a sample space, event and Boolean algebra, the next remaining element to complete Consistent Histories approach is Probability Distribution. Consistent Histories approach allows the assignment of probabilities to a family of histories of a system if and only if the family is consistent. Thus, histories and family of history need to be defined first.

A *History* of a physical system is a sequence of events at successive times. A *Quantum History* of a physical system is then a sequence of *quantum* events at successive times. A *Quantum Event* at any particular time can be any quantum property of the system in question [C21].

The system has its own Hilbert Space for each quantum event, and thus *History Hilbert Space* is obtained by Tensor Product of  $f$  spaces for different histories of a single quantum system at  $f$  successive times.

$$\mathcal{H} = \mathcal{H}_1 \odot \mathcal{H}_2 \odot \mathcal{H}_3 \odot \dots \odot \mathcal{H}_f$$

The convention is to use  $\odot$ , instead of  $\otimes$ , as a symbol for temporal (multi-time), rather than spatial (multi-partite) composites.

Each time on the history has a quantum event associated with it, which is represented by its projector  $F_i$  for time  $t_i$ , and the temporal tensor product of these projectors makes up a history.

$$Y = F_1 \odot F_2 \odot F_3 \odot \dots \odot F_f$$

The conventional notation when one has projectors in dyad form,  $|x\rangle\langle x|$  is to write them as  $[x]$ .

For a single time, a sample space of histories is a decomposition of the identity on the History Hilbert Space,  $I = \sum_{\alpha} Y^{\alpha}$ . The Boolean event algebra for such a sample space is referred to as the *Family of Histories* [C21]:

$$Y = \sum_{\alpha} \pi_{\alpha} Y^{\alpha}.$$

If the projectors corresponding to the properties do not commute with one another, and are thus not orthogonal, the properties are incompatible [C21]. This incompatibility means that these properties cannot be logically compared, which is a hallmark of quantum physics, and does not arise in its

classical counterpart. Thus, for consistent quantum reasoning, using a *single framework* is a necessity.

A framework is a Boolean algebra of commuting projectors based upon a suitable sample space. Different quantum frameworks are generally incompatible, which means that the corresponding descriptions cannot be combined. Therefore, when constructing a quantum description of a physical system it is necessary to restrict oneself to a single framework, or in the very least, not mix results from incompatible frameworks. This is referred to as the *single framework rule* [C21].

## 5.4 Assigning Probability to Histories

The time development operator that governs the evolution of the system is given as  $T(t', t)$ , such that  $|\psi(t')\rangle = T(t', t)|\psi(t)\rangle$ . The probability then comes from the Born rule:

$$P(\psi(t')) = |\langle\psi(t')|\psi(t)\rangle|^2.$$

Assigning probabilities to histories is simply an extension of the Born Rule.

Let's take a general history as below:

$$Y = F_1 \odot F_2 \odot F_3 \odot \dots \odot F_f$$

The probability of a history, called the *weight of a chain*, is given as:

$$W(Y) = |\langle F_f | F_1 \rangle|^2$$

coming from the states corresponding to the final and first projectors. To simplify what will be an expansion of the above formula for the weight, chain operators are defined. The chain operator  $K(Y)$  for the history  $Y$  is defined as

$$K(Y) = F_f T(t_f, t_{f-1}) F_{f-1} T(t_{f-1}, t_{f-2}) F_{f-2} \dots F_2 T(t_2, t_1) F_1.$$

The conjugate transpose of the chain operator is then

$$K^\dagger(Y) = F_1 T(t_1, t_2) F_2 T(t_2, t_3) F_3 \dots F_{f-1} T(t_{f-1}, t_f) F_f.$$

Expanding the expression for the weight of the chain, one reaches the result below, where the weight is computed as the inner product from the chain operators as:

$$W(Y) = \langle K^\dagger(Y) | K(Y) \rangle = \text{Tr}[K^\dagger(Y) K(Y)].$$

## 5.5 Consistency Condition

Consistency condition is a condition for a family of histories and comes from the rules of classical probability that elements of a sample space are mutually exclusive. Weights, when used to assign probabilities, are additive functions, meaning that if  $E \cap F = \emptyset$ , which would be required to be mutually exclusive,

$$W(E \cup F) = W(E) + W(F).$$

This implies that two mutually exclusive events cannot happen simultaneously. As by definition, chains of histories should be mutually exclusive, this leads to

$$\langle K^\dagger(Y^\alpha) | K(Y^\beta) \rangle = \text{Tr}[K^\dagger(Y^\alpha) K(Y^\beta)] = 0,$$

for all  $\alpha \neq \beta$ , which is called the *consistency condition* [C21].

If a family of histories satisfies consistency condition, it is called a *consistent family* or a *decoherent family*. It is important to note that consistency applies to a family of histories, and not to individual histories [C21].

A family of histories that does not satisfy the consistency conditions is called an *inconsistent family*. The consistency of a family of histories is inherently dependent on the dynamics of the system. Changing the dynamics of a system can render a consistent family inconsistent, and vice versa [C21].

How does one interpret an inconsistent family? It can be very tempting to think of an inconsistent family of histories as an event that *never occurs* (probability equals 0). However, that would be an inaccurate description of the physically deep notion of inconsistency [C31].

Individual histories with vanishing probability,  $K(M) = 0$  never occur and are called *dynamically impossible*. Whereas inconsistent families are meaningless in Consistent Histories Approach, in the sense that there is no way to assign them probabilities within the context of a stochastic time development governed by the laws of quantum dynamics [C21].

## 6 (Article A) Neutrino Oscillations in the Presence of Matter and Continuous Non-Selective Measurement

### 6.1 Introduction

The aim of the first article was to show the dissipation of properties of an open quantum system resulting from the coupling of one of the Observables of the system with a “classical” surrounding. The system was of neutrino oscillations in the presence of normal matter.

The “environment”, which was modelled as a “probe” in a measurement model, mimics a continuous non-selective measurement. Using this system and environment, the goal was not to analyze any realistic measurement scenario, but to adopt typical terminology for quantum measurements [A17,A22], and utilise it to describe the effect of a continuous non-selective measurement.

The hypothesis was that an environment mimicking a continuous non-selective measurement significantly alters neutrino inter-flavor transition probabilities, and steady-state correlation function of the neutrino’s flavor.

Article A presented a brief overview of neutrino oscillations, dynamical properties of transition between flavors, and steady-state autocorrelation of flavor observable, and analyzed them in the presence of normal matter, and decoherence [A4,A7-A12] induced by a *measurement like mechanism* that mimics continuous non-selective measurement of Observables [17].

The Observables were taken to be analogous to one of the angular momentum Observables and two such Observables were considered in the study. The first one resembled a potential flavor measurement, and the second resembled a hypothetical mass measurement of the neutrino. The article detailed the role of a scattering-like interaction between neutrinos and normal matter, CP symmetry, and its violation.

## 6.2 Model

Article A employed a qutrit approximation of neutrino oscillations. As discussed in the second chapter of the dissertation, neutrino of any given flavor state, after a propagation of some distance and time, can be observed in a different flavor state, a consequence of the non-correspondence between the flavor  $\{v_e, v_\mu, v_\tau\}$  and mass states  $\{v_1, v_2, v_3\}$  [A1,A3]. The flavor and mass states connected through Pontecorvo-Maki-Nakagawa-Sakata unitary lepton mixing matrix,  $U_{PMNS}$ , as presented in the second chapter.

Article A considered a Dirac type Neutrino. The  $U_{PMNS}$  matrix for such a case then has three mixing angles  $\{\theta_{12}, \theta_{23}, \theta_{13}\}$ , and a Charge Conjugation Parity (CP) violating phase  $\delta$  [A1]. The time evolution is governed by the Schrödinger equation in the flavor basis, Equation 3 from the Article:

$$i\frac{d}{dt}\psi_F = H_F\psi_F$$

where  $H_F$  describes the neutrino flavor state at a given time  $t$ , and  $H_F$  is the Hamiltonian [A1,A19-A21], equation 4 of the article:

$$H_F = H_{kin} + H_{pot}$$

representing the kinetic and potential parts of the Hamiltonian. The  $H_F$  takes the form as stated in the second chapter for the qutrit approximation of neutrino oscillation, which make up the equations 5 and 6 of the article.

The Environment is modelled to mimic the action of a Continuous Non-Selective Measurement, as detailed in the third chapter of this dissertation. This leads to dynamics that is no longer unitary [A24], resulting in quantum master equation [A25], equation 9 from the article:

$$\frac{d}{dt}\rho(t) = -i[H_{sys}, \rho(t)] - \frac{\sigma}{2}[A, [A, \rho(t)]]$$

$A$  is the observable coupled with the environment, and  $\sigma$  describes the back-action of such a measurement, given as equation 10 of the article. The more accurately the observable  $A$  is measured, the more information is gained, the larger the value of  $\sigma$  [A25]. If one identifies  $t_z = \frac{1}{\sigma^2}$  as Zeno time, as  $t_z \rightarrow 0$ , one gets the quantum Zeno effect [A25,A26].

The Hamiltonian for the system for this master equation was taken to the Hamiltonian for the neutrino oscillations, that is  $H_{sys} = H_F$ , and the two

Observables that couple with the system were taken to be  $F$  and  $M$ , expressed as equations 12 and 13 of the article respectively.

$$F = \sum_{x=e,\mu,\tau} a_x \Pi_x, \text{ where } \Pi_x = |v_x\rangle\langle v_x|, x = e, \mu, \tau$$

$$M = \sum_{x=1,2,3} a_x \Pi_x, \text{ where } \Pi_x = |v_x\rangle\langle v_x|, x = 1, 2, 3.$$

The article did not claim that any measurement schemes for the above ‘Observables’ are currently experimentally accessible, especially mass-state measurement. It simply describes a back-effect of a potential non-selective measurement-like quantum-classical environment affecting neutrino oscillation.

Firstly, the probability of the transition of an electron neutrino into one of three flavors was computed. For each transition, three cases were compared, (a) free oscillation in an absence of measurement, (b) the ‘flavor measurement’ of the observable  $F$ , and (c) the ‘mass-state measurement’ given by  $M$ .

Secondly, the study focused on the effect of continuous non-selective measurement on the two-time steady state correlation function of observable  $F$ ,  $\langle F(t+K)F(t) \rangle_{ss}$  [A28].

### 6.3 Results

Following the results from [A18], the following parameters were set in Article A:

- $s_{12}^2 = 0.307$
- $s_{13}^2 = 0.021$
- $s_{23}^2 = 0.5$
- $\Delta m_{12}^2 = 7.37 \cdot 10^{-5} eV^2$
- $\Delta m_{31}^2 = 2.52 \cdot 10^{-3} eV^2$

The initial state of the neutrino was fixed to that of an electron neutrino and amplitude of back-action as  $\sigma = 0.1$ . For numerical calculations, the Master equation solver from the Python-based toolbox QuTip [A29,A30] was utilized. Particularly, `qutip.mesolve` module was used to obtain numerical solutions to calculate flavor transition probabilities, and `correlation_2op_1t` with a parameter `state0=None` for steady-state autocorrelation function.

Starting with flavor transition probabilities, the expectation is that neutrinos evolving under such dynamics approach equilibrium, that is equal transition probabilities:  $P_{e \rightarrow x} = \frac{1}{3}$  for each  $x = e, \mu, \tau$ . The article, however, delved into the results observed for short-time transitions (compared to equilibrium time), presented for five sets of values for  $V_C$ , and time  $t$  scaled to units of  $E$ .

In the absence of normal matter, and a vanishing CP-violating phase, Continuous non-selective measurement of both  $F$  and  $M$  led to a significant enhancement of electron to muon neutrino probability. The remaining inter-flavor transitions became less likely to be measured as shown in figure 1.

For a vanishing CP-violating phase, but in the presence of normal matter, the article showed suppression of probability for the oscillation, and concluded that continuous non-selective measurement of neutrino's flavor (or mass) can be responsible for a potential suppression of neutrino oscillations, as evidenced in figure 2.

Figures 3 and 4 explored two sets of values for a non-vanishing CP-violating phase in the absence of normal matter. Significant enhancement of electron to muon neutrino probability was observed, more so for "mass-state measurement". The amplitude of the remaining inter-flavor transitions suppressed relatively faster.

Figure 5 portrayed the probability of flavor transitions in the presence of normal matter and a non-vanishing CP-violating phase. Further enhancement of electron to muon neutrino probability was seen, more so for "mass-state measurement".

Figure 6 presented the effect of measuring the mass and flavor Observables for vanishing CP-violation phase on the Steady-State autocorrelation function. A comparison was drawn between the absence and presence of normal matter, that is a vanishing and non-vanishing  $V_C$ . It was noticed that in the presence of normal matter, the steady-state correlation functions became

less sensitive to the choice of the observable being measured (either  $F$  or  $M$ ).

This effect was even more pronounced in Figure 7, which had non-vanishing CP-violating phase and charge-current potential  $V_C$ , which made the autocorrelation  $\langle F(t)F(0) \rangle$  for both measurement Observables harder to differentiate.

## 6.4 Conclusion

The article described the dynamics of neutrino oscillation in the presence of an environment that mimicked continuous non-selective measurement. It showed significant impact of the presence of normal matter, indicated by non-vanishing charge current potential, and a possible CP-violation. The probability of transition from electron flavor to muon was enhanced, and even more significantly for CP-violating systems. For the two remaining transitions, electron to electron, and electron to tau, continuous non-selective measurement led to the damping of oscillations, which was particularly strong for the systems coupled ('measured') via  $M$  to the environment (probe).

Autocorrelation of the 'flavor observable'  $F$  was the second quantifier studied in the article, though it is not commonly done so for neutrinos. Continuous non-selective measurement led to the damping of oscillations in the presence of normal matter and CP-violation, rendering it indistinguishable for  $F$  and  $M$  for times shorter than the equilibrium time.

Selective measurements of neutrinos are extremely rare and difficult, but the model in this article incorporated a 'close to non-selective' case of it, which helped shed light on dynamical properties of neutrinos and the role played by fundamental symmetries, like the CP-violation.

## Article

# Neutrino Oscillations in the Presence of Matter and Continuous Non-Selective Measurement

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**Abstract:** We investigate three-flavor neutrino oscillation affected by an environment mimicking a continuous non-selective measurement. We show that such a coupling that is given by a measured observable affects probability of inter-flavor neutrino transition and a steady-state correlation function of the neutrino's flavor. We juxtapose and compare our predictions influenced by matter's scattering and *CP*-violation.

**Keywords:** neutrino oscillation; quantum open systems; continuous measurement

## 1. Introduction

Time-evolving neutrinos [1] attract attention and stimulate research from very different perspectives [2,3]. Because of very peculiar and intriguing dynamic properties of neutrino oscillations [4–6], the most natural particle physics context becomes supplemented by various studies carried in seemingly far research communities such as decoherence [4,7–12] or quantum information [5,13–16]. In this paper, after short review of neutrino oscillation phenomenon, we discuss dynamic properties of inter-flavor transitions and steady-state autocorrelation of neutrino oscillation in a presence of matter and decoherence induced by a measurement-like mechanism effectively identical to a continuous non-selective measurement [17] of two 'observables' analogous to one of the angular momentum observables: the first corresponds to a potential flavor measurement and the second, very hypothetical, to the measurement of neutrino mass. We study dynamical properties of this model of neutrino oscillation and show that it becomes equipped with a features significantly altering inter-flavor transitions probabilities and steady-state autocorrelation of the 'flavor observable'. We investigate, in detail, the role of scattering-like interaction between neutrinos and normal matter *CP*-symmetry and its violation.

### 1.1. Neutrino Oscillation of Three Flavors

Neutrino oscillation occurs when the neutrino of a given flavor  $\alpha = \{e, \mu, \tau\}$  can be after some distance and some time measured in a different flavor state. It is well known [1,3] that it is a consequence of the (non)correspondence between flavor  $\{\nu_e, \nu_\mu, \nu_\tau\}$  and massive neutrino states  $\{\nu_1, \nu_2, \nu_3\}$ , which are connected via Pontecorvo-Maki-Nakagawa-Sakata unitary lepton mixing matrix  $U_{PMNS}$ :

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{PMNS} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad (1)$$

where the matrix  $U_{PMNS}$  in the case of Dirac neutrino (considered in this paper) is parameterized with three mixing angles  $\theta_{12}$ ,  $\theta_{23}$ ,  $\theta_{13}$  and one CP (charge conjugation parity)-violating phase  $\delta$  [1]:

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (2)$$

with  $c_{ij} = \cos(\theta_{ij})$  and  $s_{ij} = \sin(\theta_{ij})$  (following Ref. [18] we set  $s_{12}^2 = 0.307$ ,  $s_{13}^2 = 0.021$  and  $s_{23}^2 = 0.5$ ) and  $\delta$  being the CP-violating phase. The results obtained at the current level of experimental sensitivity are compatible with  $\delta = 0$ ; however, as shown below, the hypothetical CP-violation and the (non-vanishing) phase  $\delta$  is a highly non-trivial parameter affecting neutrino oscillation.

The time evolution of the neutrino is governed by the Schrödinger equation in the flavor basis:

$$i \frac{d}{dt} \Psi_F = \mathcal{H}_F \Psi_F, \quad (3)$$

where  $\Psi_F$  describes the flavor neutrino state for a given time  $t$  and  $\mathcal{H}_F$  is the Hamiltonian [1,19–21]:

$$\mathcal{H}_F = H_{kin} + H_{pot}, \quad (4)$$

consisting of the kinetic  $H_{kin}$  and  $H_{pot}$  potential parts to be described below.

Taking into account an ultrarelativistic limit for the neutrinos (with neutrino's eigenenergies  $E_k = E + \frac{m_k^2}{2E}$  and momentum  $p \sim E$ )—the kinetic part of the Hamiltonian in the flavor basis reads:

$$H_{kin} = \frac{1}{2E} U_{PMNS} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} U_{PMNS}^\dagger, \quad (5)$$

where  $\Delta m_{ij}^2 = m_i^2 - m_j^2$  is the difference between the masses of two oscillating neutrinos. In the following numerical calculations, we set  $\Delta m_{12}^2 = 7.37 \cdot 10^{-5} eV^2$  and  $\Delta m_{31}^2 = 2.52 \cdot 10^{-3} eV^2$  [18].

For the potential part  $H_{pot}$ , the main contribution originates from neutrino's interaction (via the coherent forward elastic scattering and for negligible non-coherent effects) with the matter's electrons and the weak charge of the nuclei generating charged-current potential  $V_{CC}$ . The potential part of the Hamiltonian takes then the following form:

$$H_{pot} = \begin{pmatrix} V_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (6)$$

The charged-current potential  $V_C$  is related to the electron's density in the matter  $V_C = \sqrt{2}G_F n_e$  with the Fermi coupling constant  $G_F$  and the electron number density  $n_e$ .

## 1.2. Continuous Non-Selective Measurement

Projective ideal measurement originating from famous von Neumann ideas plays significant role for understanding fundamental differences between classical and quantum models of Nature. However, it is well known that real experiments that are associated with measuring physical quantities almost never can be sufficiently described in terms of von-Neumann scheme [17,22,23]. The reason is easy to express: one can hardly perform a measurement on a quantum system *per se*. One rather attempts to allow the system under consideration to interact with some other system—a probe—which properties are accessible for classical apparatus. The indirect measurement can be summarized in a

following two-step procedure: First, (i) an observable  $A$  of a quantum system to be measured couples to a probe ‘position’  $Q$  resulting in a term  $H_{int} = A \otimes Q$  in a system-probe total Hamiltonian

$$H_{sys+probe} = H_{sys} + H_{probe} + g(t)H_{int} \quad (7)$$

Subsequently, (ii) the probe ‘momentum’  $P$  canonically conjugated to  $Q$  is (projectively) measured.

If one assumes non-selective measurement—a measurement with the result ignored—carried on a time interval subdivided into intervals of length  $q$ , the reduced (with respect to the probe) dynamics of the quantum system is no longer unitary, but rather becomes a quantum completely positive map [24].

Let us recall that a map of a Hilbert space  $\Lambda : \mathcal{H} \rightarrow \mathcal{H}$  is completely positive if its (trivial) extension  $\Lambda \otimes \mathcal{I}_{n \times n} : \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K}$  with an identity  $\mathcal{I}_{n \times n}$  of an arbitrary dimension  $n$  maps quantum states (positive operators) into quantum states. The concept of complete positivity is one of the possible guidelines for constructing reduced non-unitary dynamics  $\Lambda_t : \mathcal{H}_{sys} \rightarrow \mathcal{H}_{sys}$  of composite  $\mathcal{H}_{sys} \otimes \mathcal{H}_{probe}$  quantum systems. In particular, for a set of measurement operators  $E_x$  (such that  $\sum_x E_x^\dagger E_x = I$ ), the non-selective measurement of a system in state  $\rho$  results in a transformation  $\rho \rightarrow \sum_x E_x \rho E_x^\dagger$  and is an example of complete positive map [22]. In the case considered here, the measurement on each time (sub)interval  $q$  results in

$$\rho(t+q) = \int dp E_p \rho(t) E_p^\dagger \quad (8)$$

with  $E_p = e^{iqH_{sys}} \langle p | \exp(-iqH_{int}) | \phi_q \rangle$ , where  $|p\rangle \in \mathcal{H}_{probe}$  is an eigenstate of  $P$  and  $|\phi_q\rangle \in \mathcal{H}_{probe}$  is a state of the probe before measurement. For a finite-time duration of a measurement, expanding Equation (8) to the second order in  $q$ , one arrives to the corresponding dynamics of a system density matrix generated by a quantum master equation [25]

$$\frac{d}{dt}\rho(t) = -i[H_{sys}, \rho(t)] - \frac{\sigma}{2} [A, [A, \rho(t)]] \quad (9)$$

with a parameter  $\sigma$  describing back-action of a measurement. For an integrated coupling strength  $q \sim \int_0^t g(s)ds$  proportional to the length of time (sub)interval  $q$  the parameter  $\sigma$  reads as

$$\sigma^2 := \lim_{q \rightarrow 0} q \langle \phi_q | Q^2 | \phi_q \rangle \quad (10)$$

where the limit describes passage from a step-like to a truly continuous measurement. As the measurement involves conjugate variables  $p, Q$  of the probe, the more accurately the observable  $A$  is measured, i.e., the more information is gained, the larger  $\sigma$  enters Equation (9), as discussed in Ref. [25]. Let us notice that if one identifies  $t_z = 1/\sigma^2$  with a Zeno time, in the limit  $t_z \rightarrow 0$  one arrives to the celebrated quantum Zeno effect [25,26]. In other words, the case of finite  $\sigma$  applied here can be viewed as a ‘non-ideal Zeno effect’.

Further, we consider continuous non-selective measurement of a neutrino described by the master equation Equation (9), with

$$H_{sys} = \mathcal{H}_{\mathcal{F}} \quad (11)$$

and we limit our attention to two observables. The first

$$F = \sum_{x=e,\mu,\tau} a_x \Pi_x, \text{ where } \Pi_x = |\nu_x\rangle \langle \nu_x|, x = e, \mu, \tau \quad (12)$$

corresponds to a ‘favour measurement’. The second, diagonal in a mass-state basis

$$M = \sum_{x=1,2,3} a_x \Pi_x, \text{ where } \Pi_x = |\nu_x\rangle \langle \nu_x|, x = 1, 2, 3 \quad (13)$$

will be called a ‘mass-state measurement’. For both observables  $F, M$ , we assume  $a_e = -a_\tau = 1$  and  $a_\mu = 0$  in analogy to standard representation of one of the angular momentum operators  $J_z \sim \text{diag}(-1, 0, 1)$ . In other words, we consider two unitary coupled observables  $F \sim U_{PMNS} M U_{PMNS}^\dagger$  formally representing a  $J_z$ -component of angular momentum in two unitary equivalent bases: the flavor  $\{|\nu_x\rangle\}_{x=e,\mu,\tau}$  and the mass-state  $\{|\nu_x\rangle\}_{x=1,2,3}$  basis, respectively.

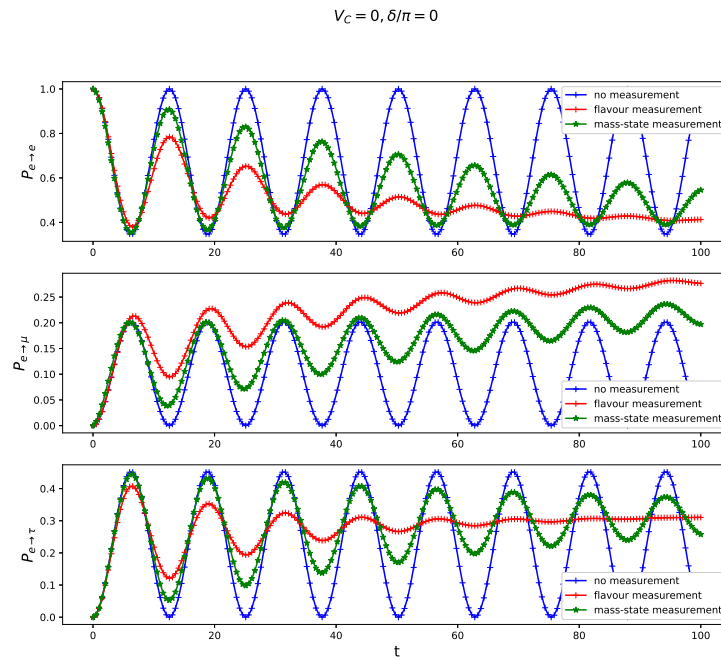
Let us emphasise that Equation (9), if compared e.g., to a so called singular coupling limit [27], formally describes an open system with a particular, measurement-originated coupling  $H_{int}$ . To be precise, we are far from a claim that any measurement schemes for the above ‘observables’ are currently experimentally accessible: in particular, it is clearly and obviously the case of the mass-state measurement, which is known to be unavailable for neutrinos. Our aim is to describe a back-effect of a potential non-selective measurement-like quantum-classical environment affecting neutrino oscillation (i.e., a very particular type of decoherence) rather than to analyse any nowadays realistic measurement scenario we simply adopt terminology typical for quantum measurement [17,22].

## 2. Results

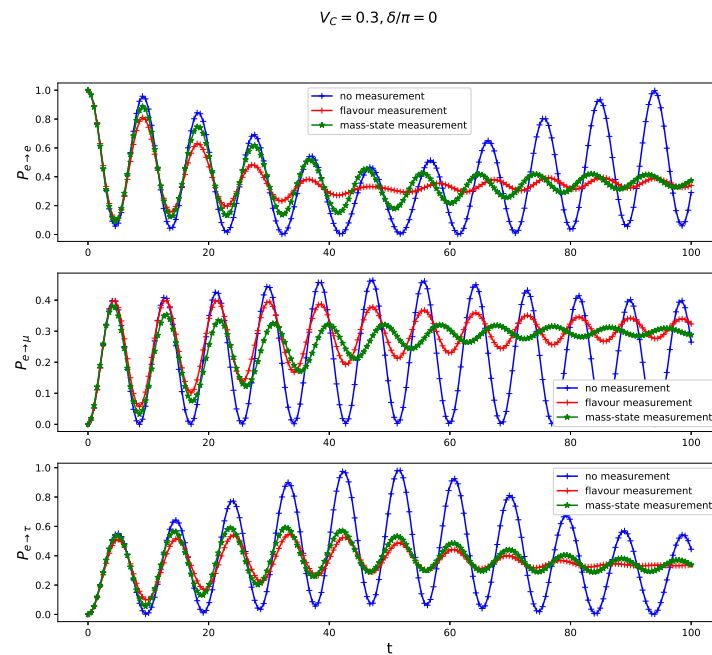
We analyse an effect of non-selective measurement on neutrino oscillation quantified by a probability  $P_{e \rightarrow x}$  (with  $x = e, \mu, \tau$ ) of a neutrino’s change of flavor, i.e., of a transition of an electron neutrino into one of three flavors  $\nu_e \rightarrow \nu_x$ . We compare three cases: (i) free oscillation in an absence of a measurement, (ii) the ‘flavor measurement’ of the observable  $F$  Equation (12), and (iii) the ‘mass-state measurement’ given by  $M$  in Equation (13).

Effective dynamics in the presence of continuous non-selective measurement leads to a Markovian master equation formally identical to class of dissipative open quantum systems resulting in continuous information leakage from the system to its ‘environment’ which in our case is a ‘probe’ in a measurement model. It is expected that neutrinos, evolving according to Equation (9), approach finally their equilibrium [24] which, although non-thermal, is characterized by equal transition probabilities  $P_{e \rightarrow x} = 1/3$  for each  $x = e, \mu, \tau$ . However, if one considers short-time transient behaviour, one observes significantly different behaviour of different probabilities as presented in Figures 1–5. Continuous non-selective measurement of both  $F$  and  $M$  leads to a significant enhancement of  $P_{e \rightarrow \mu}$  probability in comparison with remaining inter-flavor transitions which become less likely in a short-time evolution as presented in Figure 1. Moreover, continuous measurements, either of  $F$  or  $M$ , result in the suppression of probability oscillations which becomes particularly visible in a presence of non-vanishing charge-current potential, as presented in Figure 2. In other words, continuous non-selective measurement of neutrino’s flavor (or mass) can be responsible for a potential suppression of neutrino oscillations.

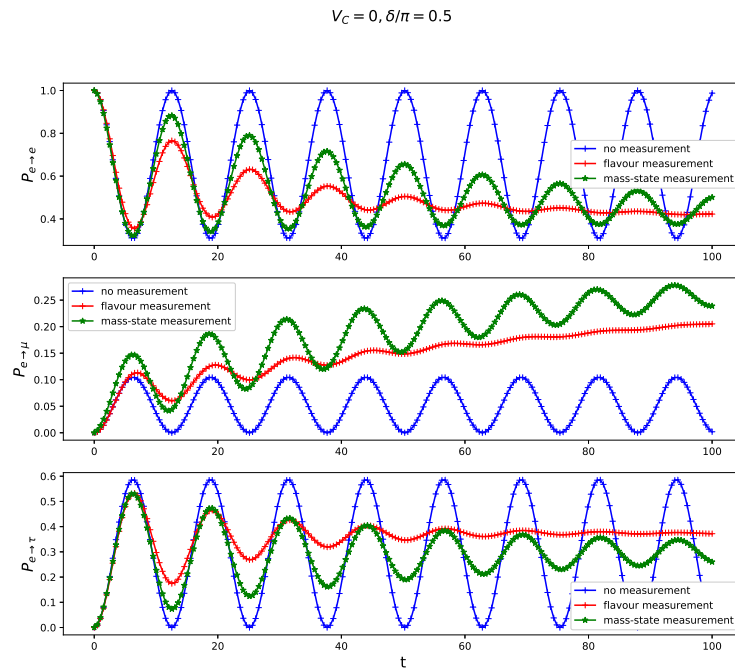
For sake of generality, we have assumed, in the model formulation, a not vanishing  $\delta \neq 0$  in Equation (2), even though the resulting CP-violation is so far entirely hypothetical. We present in Figure 3 that the most significant impact of the CP-violation indicated by  $\delta \neq 0$  occurs for the  $\nu_e \rightarrow \nu_\mu$  transition quantified by  $P_{e \rightarrow \mu}$ . In particular, let us notice that  $P_{e \rightarrow \mu}$  in the presence of ‘mass-state measurement’ becomes larger in comparison to ‘flavor measurement’ case for CP-violating systems what is indicated by reversed ordering of plots in central sub-panels in Figures 1, 3 and 4. This observation holds true also in the presence of charge-current potential  $V_C$ , as can be inferred from comparison central panels of Figures 2 and 5.



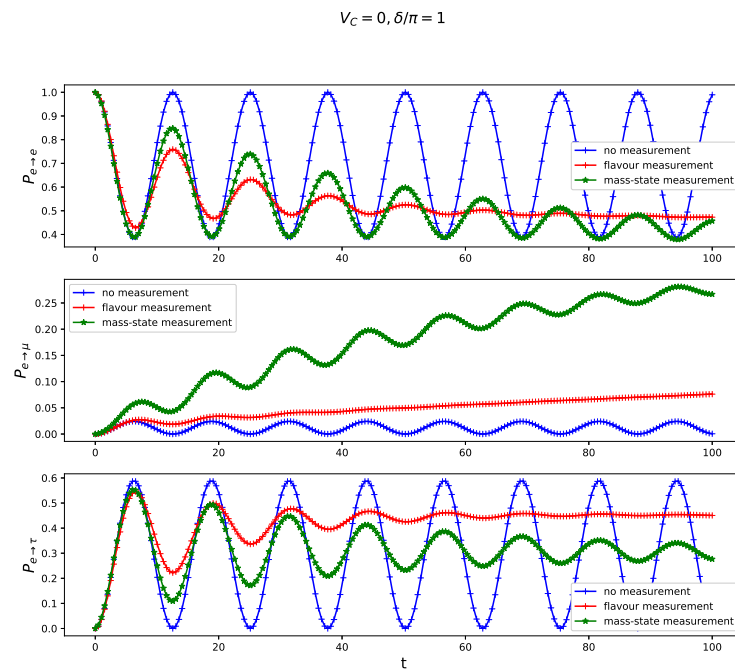
**Figure 1.** Transition probability for free three-flavor neutrino oscillation juxtaposed with the oscillations modified by a continuous non-selective measurement of the flavor cf. Equation (12) and ‘mass-state’ cf. Equation (13) calculated for vanishing CP-violating phase  $\delta = 0$  in the absence of normal matter  $V_C = 0$ . The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and time  $t$  is scaled in units of  $E$ .



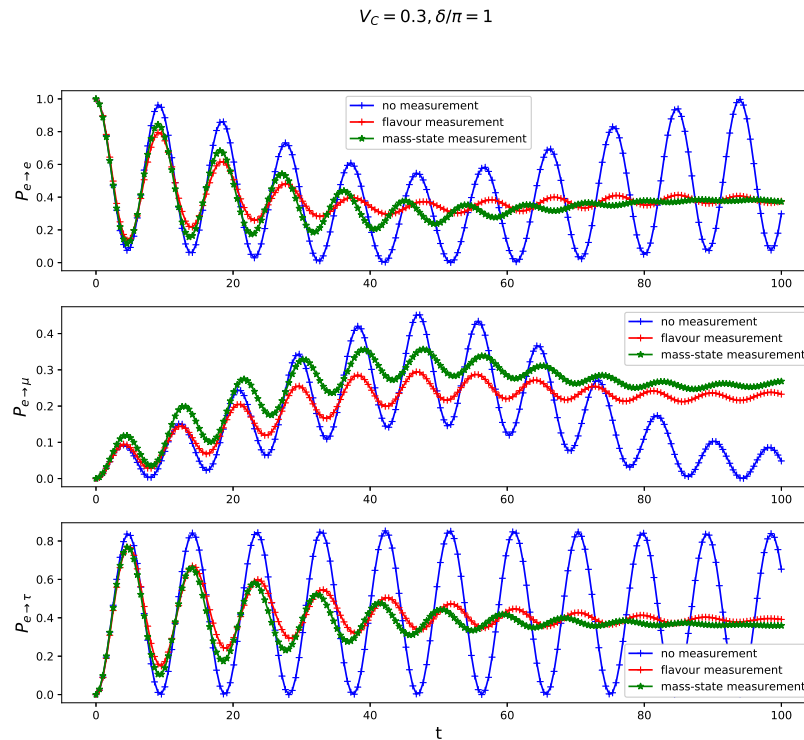
**Figure 2.** Transition probability for free three-flavor neutrino oscillation juxtaposed with the oscillations modified by a continuous non-selective measurement of the flavor cf. Equation (12) and ‘mass-state’ cf. Equation (13) calculated for vanishing CP-violating phase  $\delta = 0$  in the presence of normal matter indicated by non-vanishing charge current  $V_C = 0.3$ . The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and time  $t$  is scaled in units of  $E$ .



**Figure 3.** Transition probability for free three-flavor neutrino oscillation juxtaposed with the oscillations modified by a continuous non-selective measurement of the flavor cf. Equation (12) and ‘mass-state’ cf. Equation (13) calculated for non-vanishing CP-violating phase  $\delta = \pi/2$  and vanishing charge current  $V_C = 0$ . The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and the time is scaled in units of  $E$ .

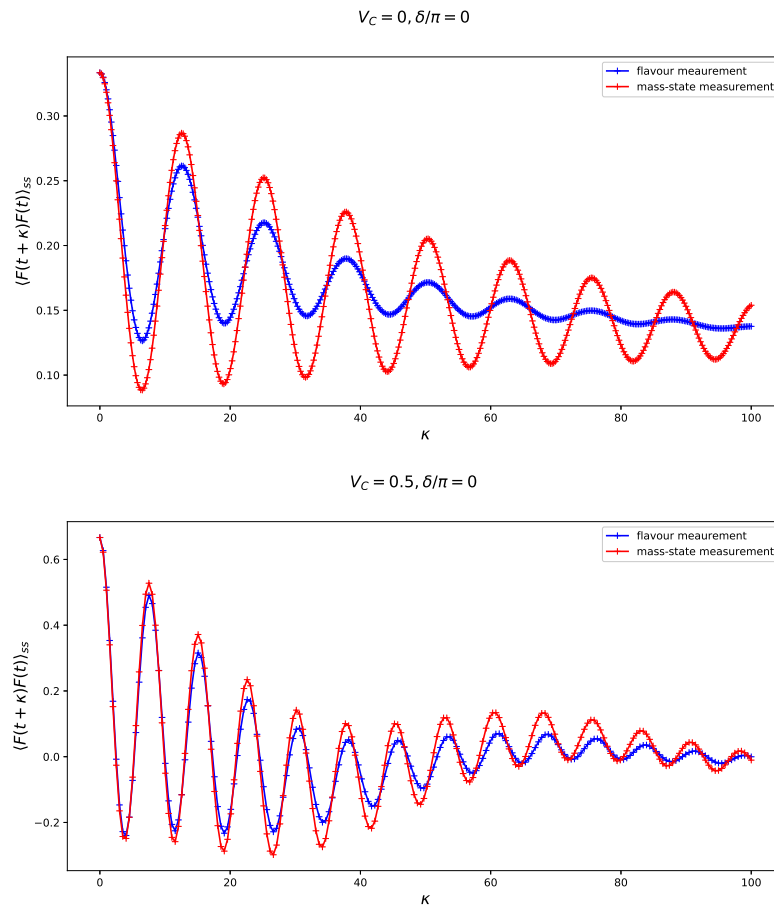


**Figure 4.** Transition probability for free three-flavor neutrino oscillation juxtaposed with the oscillations modified by a continuous non-selective measurement of the flavor cf. Equation (12) and ‘mass-state’ cf. Equation (13) calculated for non-vanishing CP-violating phase  $\delta = \pi$  and vanishing charge current  $V_C = 0$ . The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and the time is scaled in units of  $E$ .

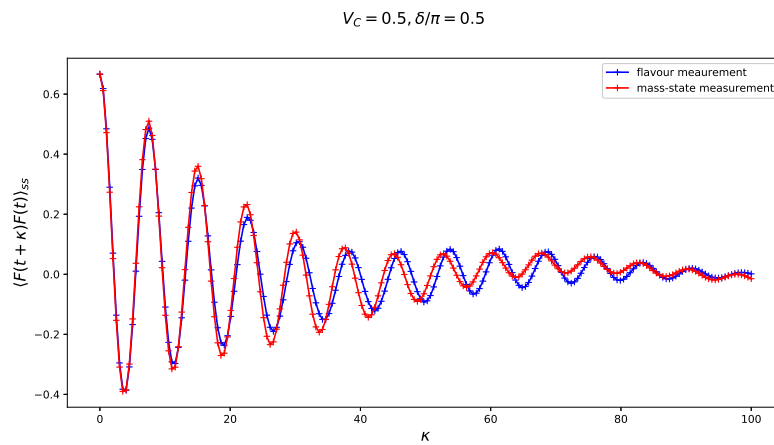


**Figure 5.** Transition probability for free three-flavor neutrino oscillation juxtaposed with the oscillations modified by a continuous non-selective measurement of the flavor cf. Equation (12) and ‘mass-state’ cf. Equation (13) calculated for non-vanishing CP-violating phase  $\delta = \pi$  and the charge current  $V_C = 0.3$ . The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and the time is scaled in units of  $E$ .

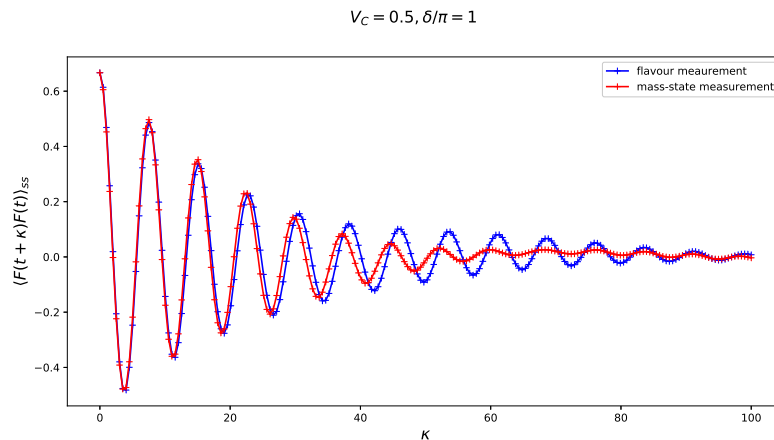
The two-time steady state correlation function  $\langle F(t+K)F(t) \rangle_{ss}$  is one of the simplest quantifiers of a multi-time structure of quantum Markov evolution [28]. Here, we study an effect of continuous measurement—either of the ‘mass-state observable’  $M$  or the flavor observable  $F$ —on the properties of the steady-state auto-correlation of  $F$ . In Figure 6 we juxtapose effects of this measurements in an absence of CP-violation for vanishing (upper panel of Figure 6 and a non-vanishing (lower panel of Figure 6 charge-current potential  $V_C$ . Let us notice that in a presence of normal matter the steady-state correlation functions become less sensitive for a choice of the observable (either  $F$  or  $M$ ) that is measured. This effect becomes even more apparent, cf. Figure 7, in a system with both non-vanishing CP-violating phase  $\delta \neq 0$  and charge-current potential  $V_C \neq 0$ , which makes the autocorrelation  $\langle F(t)F(0) \rangle$  for both measurement observables harder to distinguish.



**Figure 6.** Steady-state autocorrelation function for observable  $F$  in Equation (12) affected by a continuous non-selective measurement of the flavor and ‘mass-state’ in an absence of normal matter  $V_C = 0$  and with vanishing CP-violating phase  $\delta = 0$  (upper panel) juxtaposed with results for non-vanishing  $V_C = 0.5$  (lower panel). The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and the time is scaled in units of  $E$ .



**Figure 7.** Cont.



**Figure 7.** Steady-state autocorrelation function for observable  $F$  in Equation (12) affected by a continuous non-selective measurement of the flavor and ‘mass-state’ in the presence of normal matter  $V_C = 0.5$  and with non-vanishing CP-violating phase  $\delta = \pi/2$  (upper panel) juxtaposed with results for  $\delta = \pi$  (lower panel). The amplitude of a back-action in Equation (9)  $\sigma = 0.1$  and the time is scaled in units of  $E$ .

### 3. Discussion

There are various sources of dissipation affecting quantum dynamics in various time scales: one can study thermal or non-thermal decoherence, asymptotic equilibrium or short time transient dynamics of a system far from equilibrium. Our results concern very special source of dissipation in neutrino oscillation that originates from the coupling of one of the observables to a classical surrounding. This model is formally equivalent to a continuous non-selective measurement. For three-flavor oscillation studied in this paper, one can distinguish two bases that are of a great physical importance: the mass state basis and the flavor basis related by the Pontecorvo-Maki-Nakagawa-Sakata unitary lepton mixing matrix Equation (1) and the related (diagonal) observables Equations (12) and (13) of formally related to the symmetry to the angular momentum operators in three dimensional representation of the  $su(2)$  algebra. We investigated short time (with respect to equilibration time) inter-flavor transitions probabilities of an initially electron neutrino. We showed significant impact of both a presence of normal matter indicated by non-vanishing charge current and the potential CP-violation. In particular, one expects an enhancement of the  $\nu_e \rightarrow \nu_\mu$  transition probability, as presented in Figures 1 and 2, an effect that becomes even stronger for CP-violating systems, as presented in Figures 3–5. For remaining transitions  $\nu_e \rightarrow \nu_{e,\tau}$ , the main effect is the suppression of oscillations which is particularly strong for the systems coupled (‘measured’) via  $M$  to an environment (probe). Autocorrelation of the ‘flavor observable’  $F$  is the second quantifier studied in this paper, although less obvious in a context of neutrinos, is a very basic for identifying non-trivial multi-time structure of quantum dynamics. In Figure 6, we present the steady-state autocorrelation for two previously studied types of ‘measurement’. Suppression of oscillations in a presence of both normal matter and CP-violation Figure 7 is here the most striking property.

### 4. Conclusions

There are many aspects of neutrino oscillation that make this phenomenon a natural object of interdisciplinary research. In particular, a three-level simplification of this otherwise highly complex physical system allows for extracting its dynamical properties in various phenomenologically modelled circumstances. Last but not least, it provides an example of a quantum dynamical system of the properties seemingly far from particle physics, closer related to quantum information and measurement theory. Our work, although speculative in character, describes the dynamics of neutrino oscillation

in the presence of an environment that mimics continuous non-selective measurement and relates a potential suppression of neutrino oscillation to the (imperfect) Zeno-like mechanism. Selective measurements of neutrinos are extremely difficult and rare, we hope that our model incorporating the ‘almost successful’ non-selective case sheds light on dynamical properties of neutrinos and the role played by fundamental symmetries, such as the celebrated CP-violation.

## 5. Materials and Methods

For numeric calculations we use the Master equation solver from the Python-based toolbox *QuTip* [29,30]. In particular, we use `qutip.mesolve` module to obtain numerical solution (quantum state) of Equation (9) to calculate probabilities  $P_{e \rightarrow x}$ ,  $x = e, \mu, \tau$ . We use `correlation_2op_1t` with a parameter `state0=None` for a steady-state autocorrelation function.

**Author Contributions:** Conceptualization, F.K. and J.D.; methodology, J.D.; software, F.K. and J.D.; formal analysis, F.K. and J.D.; writing—original draft preparation, J.D.; writing—review and editing, F.K. and J.D. All authors have read and agree to the published version of the manuscript.

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## 7 (Article B) Geometric speed limit of neutrino oscillation

### 7.1 Introduction

The aim of the second article was to examine the effects of the environment and decoherence on the geometric speed limit of an evolving system. For continuity, the system taken for the study was neutrino oscillations in the presence of normal matter. Furthermore, the article explored the impact of a hypothetical CP-violation on geometric speed limit and discussed if geometric speed limit can be used as a tool to quantify the said CP-violation.

As detailed in the fourth chapter, it was the realization by Mandelstam and Tamm [B42] that  $\Delta t$  in the Energy-Time Uncertainty Relation can be seen as the time a quantum system needs to evolve from an initial state to a final state, which was formalized as the Quantum Speed Limit (QSL) Time,  $\tau_{QSL}$ .

Fidelity Function  $F$ , can be defined as a measure of closeness of two quantum states. If both states are pure, Fidelity equals the transition probability between the states [B13]. The root of the fidelity function, referred to as Root Fidelity, features in the formulation of Bures Angle, a measure of the statistical distance between quantum states [B46]. As expanded in the fourth chapter of the dissertation, the Geometric QSL is founded on Bures angle. The Geometric QSL allows to identify how a rate of information, given by a change of statistical distance, is related to dynamical properties of quantum dynamical systems [B29].

The article considered neutrino oscillations in the presence of normal matter, and thus it was subject to decoherence effects due to the system's interaction with its environment [B10,B12], leading to non-unitary dynamics for its time evolution [B20]. As this was an open system, its evolution was determined by a reduced (with respect to its environment) density matrix [B4].

Along with the Geometric QSL, the study also computed the overlap between the flavor state during the evolution and the initial state of electron neutrino flavor. This probability was calculated as

$$P_0 = Tr[\rho_F(0)\rho_F(t)].$$

The article investigated the results for overlap  $P_0$  and the Geometric QSL for the three cases, interaction with normal matter, decoherence, and non-vanishing CP-violating phase.

## 7.2 Model

Following [B10,B12], the article neglected non-Markovian effects in the evolution of the open system of neutrino propagating in normal matter, leading to evolution being governed by Gorini–Kossakowski–Sudarshan–Lindblad Markovian master equation [B4,B20]:

$$\frac{d}{dt}\rho_F(t) = \mathbb{L}[\rho_F(t)],$$

where  $\rho_F \in \mathcal{B}(\mathbb{C}^3)$  is a neutrino flavor state. The generator  $\mathbb{L}[\cdot]$  consists of the Hamiltonian and the Lindbladian parts.

The Geometric QSL is then given as

$$\nu_{QSL}(t) = \frac{\|\mathbb{L}[\rho_F(t)]\|}{2\cos(l(t))\sin(l(t))},$$

where the operator norm  $\|\mathbb{L}\| = \sigma_{max}$ ,  $\sigma_{max}$  being the largest singular value of  $\|\mathbb{L}[\rho_F(t)]\|$ .  $l(t)$  is the Bures Angle, equation 3 of the article.

Just like Article A, Article B follows qutrit approximation for neutrino oscillation, resulting from the lack of correspondence between the neutrino's three flavor and mass states, related to each other through  $U_{PMNS}$  matrix [B35], comprising of the three mixing angles, and the CP-violating phase. As per experiments [B54], this phase can be non-vanishing, which was an important aspect considered in this study.

Neutrino oscillations are taken to be in the presence of normal matter. Unlike Article A, Article B does not have non-selective measurement involved. For the dynamics, the article employed complete positivity in tandem with semi-group property [B4,B12,B20]. This take us to the master equation below, which is equation 6 of the article:

$$\frac{d}{dt}\rho_F = -i[H_f, \rho_F] + L[\rho_F].$$

$H_F$  is the Hamiltonian for the system, which is the sum of the kinetic and potential parts for the qutrit approximation stated in chapter two. The Lindbladian  $L$  describes the decoherence corrections and is non-Hamiltonian.

$$L[\rho_F] = \sum_{i,j=0}^{N^2-1} c_{ij} \left( F_i \rho_F F_j^\dagger - \frac{1}{2} \{F_j^\dagger F_i, \rho_F\} \right).$$

$N$  is equal to the dimension of the system, which is 3 in this case.  $\{\cdot\}$  is the anti-commutator. The matrices  $F_N$  stand for generators of  $SU(3)$ , which is an 8-dimensional simple Lie group of all  $3 \times 3$  unitary matrices with determinant equal to 1. In standard representation, they are referred to as the Gell-Mann Matrices [B37], which are essentially the  $SU(3)$  analog of the Pauli Matrices for  $SU(2)$ .

$$F_N = \frac{\lambda_N}{2} \text{ where } \lambda_N \text{ are}$$

$$\begin{aligned} \lambda_0 = \mathbb{I}_3, \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned}$$

The coefficients  $c_{ij}$  satisfy the set of inequalities to get complete positivity [B11,B12].

$$|c_{ij}| \leq \frac{1}{2}(c_{ii} + c_{jj}).$$

The article simplified this further than just the limitations placed on these coefficients to obtain complete positivity, and considered only diagonal decoherence:

$$c_{ii} = K \text{ and } c_{i \neq j} = 0$$

### 7.3 Results

Following the results from [B18], the following parameters were set in Article B:

- $s_{12}^2 = 0.307$
- $s_{13}^2 = 0.021$
- $s_{23}^2 = 0.5$
- $\Delta m_{12}^2 = 7.37 \cdot 10^{-5} eV^2$
- $\Delta m_{31}^2 = 2.52 \cdot 10^{-3} eV^2$

The neutrino was taken to be initially in the electron flavor. Overlap  $P_0$  and the Geometric QSL were calculated through numerical solutions to the master equation, obtained using QuTip. QuTiP master equation solver *qutip.mesolve* was used for unitary and dissipative evolution of the neutrino. The solvers *qutip.bures\_angle* and *scipy.linalg.svdvals* were used to compute the Geometric QSL.

Starting with the case of normal matter, the article showed the effect of a neutrino propagating in normal matter by observing the flavor overlap probability and Geometric SQL for a set of values for charged-current potential  $V_C$ .

The presence of matter significantly affected neutrino oscillation by modifying its period as presented in the upper panel of figure 1 of the article. This was also observed in the lower panel of the figure, which was for Geometric QSL. The complete absence of normal matter,  $V_C = 0$ , the maximum values for the overlap, that is  $P_0 = 1$ , correspond with discontinuities of the Geometric QSL,  $\nu_{QSL} \rightarrow \infty$ . This agrees with the interpretation of Geometric QSL as a speed of change of statistical distance between states [B28,B29]. Due to these asymptotes, figure 1 was truncated.

Increase in  $V_C$  resulted in changes of periodicity of oscillation and the discontinuities in  $\nu_{QSL}$  disappeared and were replaced by sharp but continuous peaks. This inclusion of interaction with matter allowed to abandon the discontinuities in QSL time characteristics, which are clearly absent in real systems.

The article moved to the case of Decoherence. Numerical solution to the master equation with the Lindbladian and its coefficients was presented in the article with different sets of values of  $\kappa$  representing the effects from Decoherence along with  $V_C = 0.03$ , kept small, to avoid the discontinuities in Geometric QSL for ideal noise free systems.

A range of  $\nu_{QSL}$ , presented in figure 2 of the article, for the noise-less case,  $\kappa = 0$ , was truncated as it was irrelevant for further discussion. As per the definition of geometric speed limit, the initial values for it are asymptotic. Unlike the interaction with normal matter, Markovian Decoherence did not influence the periodicity of neutrino oscillations. It led to the damping of amplitude of the oscillations instead, as seen in the upper panel of figure 2. Decoherence of any amplitude  $\kappa$  did not alter positions of extreme values in QSL, compared to the noise-less oscillations.

Therefore, Decoherence removed discontinuities present in a noiseless time evolution of  $\nu_{QSL}$ , but did not affect the position of its peaks. For sufficiently large values of  $\kappa$ ,  $\nu_{QSL}$  underwent a qualitative change and became a monotonically decreasing function of time.

Lastly, the article presented the effects of CP symmetry and its violation on Decoherence-affected neutrino oscillation and Geometric QSL. The initial asymptotic values for  $\nu_{QSL}$  were still observed, and a range of  $\nu_{QSL}$  in the lower panel of figure 3 of the article was truncated for better visualization.

While Decoherence impacted the crests of the neutrino oscillations, CP-violation impacted the troughs of the oscillations, them being lowest for  $\delta = \pi$ . The neutrino oscillated with the same (quasi)period, but ventured further off from the initial state in the sense of Fidelity. The behavior of Geometric QSL suggested a possibility of using the relation between its amplitude and the CP-violating phase. Notably, there were time windows where both  $P_0$  and  $\nu_{QSL}$  were indistinguishable with respect to varying values of CP-violating phase. Non-vanishing  $\delta$  also enhanced peaks that were damped by Decoherence.

Thus, Geometric QSL quantified a potential CP-violation provided the time windows where both  $P_0$  and  $\nu_{QSL}$  for all values of  $\delta$  were identical were accounted for. That occurred in the vicinity of the local maxima of  $P_0$ , which corresponds with instances of time when the neutrino returns close to its initial state.

Geometric QSL has a direct relationship with quantum speed limit time [B28]. Equation 14 of the article provided the formula for this relationship as:

$$\tau_{QSL}^{-1}(t) = \frac{1}{2} \int_0^t \nu_{QSL}(t') dt',$$

and was presented in figure 4 of the article.

$V_C$  was presented in the upper panel and it influenced the magnitude and the periodicity of  $\tau_{QSL}$ , as observed for geometric speed limit. Central panel of figure 4 showed modification to amplitude only under Markovian Decoherence, as observed for geometric speed limit too. The effect of CP-violation was presented in the bottom panel of the figure, and it showed a minor, yet not insignificant, enhancement of the quantum speed limit time with the increase in value of CP-violating phase  $\delta$ .

## 7.4 Conclusion

The article considered Geometric Quantum Speed Limit and showed its potential application as a hallmark for CP-violation. It showed how QSL as a function of time becomes modified by interaction of neutrino with normal matter, and removes these “artificial” discontinuities in the QSL. Decoherence further impacts this by damping oscillations of  $\nu_{QSL}$  and for sufficiently large values of  $\kappa$ ,  $\nu_{QSL}$  behaves as a monotonically decreasing function of time. The most significant result was for the hypothetical violation of the CP symmetry for neutrinos, making it a candidate for quantifying the CP-violating phase.

From a mathematical perspective, the changes in the geometric speed limit reported are due to a collective effect of collaboration of the three external conditions applied to the system. In particular, the effect of  $\delta$  in  $\nu_{QSL}$  starts to be visible due to the presence of decoherence in a dynamical model of neutrino oscillation. Without Decoherence, that is  $\kappa = 0$ , as the numerator of  $\nu_{QSL}$  is given by maximal singular value which, to be calculated, demands multiplication of an operator and its conjugate, the CP-violating phase cancels out. The article demonstrated how purely Hamiltonian models (No Decoherence) limit the description of at least certain properties of neutrino oscillations in time.

Importantly, while all the neutrino experiments, such as the recent in [B54],

belong to the most sophisticated experiments performed so far, the predictions concerning QSL are simply qualitative signatures of certain properties of neutrino oscillations. The article discerned tailored range of time where the effect of modifications, due to presence of normal matter, decoherence, and CP-Violation, becomes most significant, not only indicating an opportunity of using QSL as a diagnostic tool for neutrino properties, for instance CP violation, but also as a tool for expressing quantum informational content of neutrino oscillation in terms of its geometric quantum speed limit.



# Geometric speed limit of neutrino oscillation

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## Abstract

We investigate geometric quantum speed limit of neutrino oscillations in a presence of matter and  $CP$ -violation. We show that periodicity in the speed limit present in an unperturbed system becomes damped by interaction with a normal matter and decoherence. We also show that (hypothetical)  $CP$ -violation causes enhancement of periodicity and increases amplitude of an oscillating quantum speed limit and can quantify  $CP$ -violation.

**Keywords** Neutrino oscillation · Quantum speed limit · Decoherence

## 1 Introduction

Time evolving neutrinos [35] and their oscillation focus and stimulate research from various perspectives. Dynamic properties of neutrino oscillations [10,11,44] initially studied in the most natural particle physics [35] context become equipped and extended by multitude of various and seemingly far investigations concerning decoherence [8,9,11,24,31,41,47] or various aspects of quantum information [5,6,10,18,33,49] to mention but a few. Such a broad interest seems to be at least partially motivated not only by natural applicability in the domain of particle physics methods and computational techniques borrowed from quantum information processing but also by recent attempts of information transfer as a resource utilizing neutrinos [52] or gravitational waves [1] reflecting an everlasting human dream of interstellar communication [25]. One can divide the quantum-information-based neutrino research into two overlap-

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ping classes: the first, focused on quantum properties of evolving neutrino—with an entanglement or contextuality as examples—and relating them to fundamental properties of these fascinating particles; the second, seemingly more exotic, motivated by neutrino-based quantum information processing. However, history of science shows that an effective feedback coupling between fundamentals and applications is a *sine qua non* condition of progress [22].

In this paper, we consider one of fundamental features of neutrino oscillation analysed as a quantum dynamical process and, after short review of neutrino time-evolution in a qutrit approximation, we discuss *geometric* quantum speed limit [29] of neutrino oscillation. Quantum speed limits (QSL) or quantum limit of evolution time reviewed in Ref. [28] can be studied from various context-dependent perspectives [7,21,27,30,34,55]: quantum control, quantum information or metrology. There were various motivations driving these investigations. The first approaches [42,43] were unified [40] and further developed in very various, sometimes divergent directions [28]. Some of them focus practical role of speed limit as a natural bound for control strategies, the other emphasize fundamental relation between various speed limits and intrinsic time scales of quantum systems [28]. In particular, contrary to recent operational approaches [21,53], geometric QSL proposed in Ref. [29] allows to identify a class of fundamental statistical and informational properties of quantum dynamics since it constitutes an upper bound for time-dependent changes of a shortest (geodesic) path qualified by the Fisher–Rao statistical distance [13] between states. With this interpretation geometric speed limit of Deffner and Lutz [29] allows to identify how a rate of information, given by a change of statistical distance, relates to dynamical properties of quantum dynamical systems governed by a very general class of equations of motion. Following [10,12], we neglect non-Markovian effects in neutrino time evolution and describe it in a qutrit approximation (i.e. as an effective qutrit using three dimensional flavour  $F$  space) in terms of Gorini–Kossakowski–Sudarshan–Lindblad Markovian master equation ( $\hbar = 1$ ) [4,20]:

$$\frac{d}{dt}\rho_F(t) = \mathbb{L}[\rho_F(t)], \quad (1)$$

where  $\rho_F(t) \in \mathcal{B}(\mathbb{C}^3)$  is a neutrino state. The generator  $\mathbb{L}[\cdot]$  consisting of Hamiltonian and Lindbladian (decohering) parts is specified in the following section. For a time-evolving quantum system Eq. (1), its geometric QSL reads [29] as follows:

$$v_{\text{QSL}}(t) = \frac{\|\mathbb{L}[\rho_F(t)]\|}{2 \cos(l(t)) \sin(l(t))} \quad (2)$$

where the operator norm  $\|\cdot\|$  is given by  $\|\mathbb{L}\| = \sigma_{\max}$  with  $\sigma_{\max}$  the largest *singular* value of  $\mathbb{L}[\rho_F(t)]$  and the Bures distance

$$l(t) = \arccos(\sqrt{F(\rho_F(0), \rho_F(t))}) \quad (3)$$

related to fidelity  $F$  [13]. The notion of QSL is based on suitably defined bound of time derivative of the Fisher–Rao metric [29]. For pure initial states  $\rho_F(0)$  Eq. (2)

is derived [28,29] using  $2 \cos(l) \sin(l) \dot{l} \leq |\text{Tr}[\dot{\rho}_F(t) \rho_F(0)]| \leq \|\mathbb{L}[\rho_F(t)]\| = \sigma_{\max}$  where the operator norm in a last inequality provides sharpest bound in comparison with trace norm or Hilbert–Schmidt [28]. The equality which relates QSL to a largest singular value  $\sigma_{\max}$  of  $\mathbb{L}[\rho_F(t)]$  is for pure initial states  $\rho_F(0)$  granted. Let us emphasize that QSL, which is a *time-dependent* quantity, characterizes quantum dynamics at a given instant contrary to other often very different operational approaches to quantum speed limits such as e.g. [53].

In this work, we investigate relations between properties of the neutrino QSL and conditions affecting neutrino oscillation: (i) interaction with normal matter’ (electrons or neutrons) induced by forward elastic scattering (in Sect. 2), (ii) an effect of decoherence resulting in nonunitary damping of neutrino oscillation (in Sect. 3). Motivated by recent experiments [54] indicating possible violation of  $CP$  (charge conjugation parity) symmetry for neutrinos we investigate (iii) the effect of  $CP$  violation relating this fundamental feature to the geometric QSL in Sect. 4. We show that the  $CP$ -violation is a highly non-trivial feature affecting QSL of neutrino oscillation being a candidate for a hallmark of  $CP$ -violation in neutrino systems. Finally, we discuss and summarize our results.

## 2 Neutrino as a qutrit

As the discovery of the neutrino oscillations, originating from mixing of *three* neutrino fields, was a first evidence in favor of a beyond the standard model physics attracting attention of particle community [14]. Following it, neutrino studies from a perspective of quantum information and communication are naturally supported by an effective description of neutrino’s flavour in terms of a qutrit approximation. It is known [35] that a neutrino of a given flavour  $\alpha \in \{e, \mu, \tau\}$  evolves in time and can be measured in a different flavour state. This phenomenon is an essence of neutrino oscillation and originates from a fundamental non-correspondence between neutrino’s flavour  $\{\nu_e, \nu_\mu, \nu_\tau\}$  and massive states  $\{\nu_1, \nu_2, \nu_3\}$ . These two sets of orthogonal states are unitarily related by Pontecorvo–Maki–Nakagawa–Sakata mixing matrix  $U_{\text{PMNS}}$ :

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{\text{PMNS}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad (4)$$

parameterized by three mixing angles  $\theta_{12}$ ,  $\theta_{23}$ ,  $\theta_{13}$  and one  $CP$ -violating phase  $\delta$  [35]:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (5)$$

with  $c_{ij} = \cos(\theta_{ij})$  and  $s_{ij} = \sin(\theta_{ij})$  (following Ref. [23] we set  $s_{12}^2 = 0.307$ ,  $s_{13}^2 = 0.021$  and  $s_{23}^2 = 0.5$ ) and  $\delta$  being the  $CP$ -violating phase. The phase  $\delta$  according to the most recent experiments [54] can be non-vanishing indicating necessity

of reformulation of most fundamental aspects of the universe at (almost) all scales. The symmetry between matter and antimatter, charge-conjugation parity-reversal  $CP$  symmetry, seemed to be a perfect symmetry of nature. However, it was proposed by Sakharov in Ref. [51] that  $CP$  violation is one of the necessary conditions for known and observed imbalance of matter and anti-matter abundance in the universe. As the  $CP$  violation known for quarks is too small to be sufficient the  $CP$  symmetry and its potential violation remains a central topic of many investigations in particle physics. In particular, results of the above mentioned experiment [54] motivate searching consequences of the neutrinos'  $CP$  violation in various areas of natural science.

Neutrino oscillations are customarily approximated in a framework of unitary evolving closed quantum systems. However, as every physical system coupled to a matter, neutrino is also subjected to decoherence effects [10,12] originating from interaction with the environment and resulting in *non-unitary* corrections to its time-evolution [20]. Decoherence effects not only serve as a possible explanation of certain experimental data [8,9,31,32,41,47]—cf. Ref. [24] for a summary of a recent progress on that topic—but also significantly modify dynamic properties of the oscillation. It is since in a presence of decoherence any system becomes *open* and its evolution is determined by a reduced (with respect to its environment) density matrix [4]. Description of a time evolution of an open system cannot be arbitrary—complete positivity [4,12,20] together with a semi-group property are two of the best known requirements.

Neutrino oscillation in a qutrit approximation Eq. (1) is given by a solution of a master equation

$$\frac{d}{dt}\rho_F = -i[\mathcal{H}_{\mathcal{F}}, \rho_F] + L[\rho_F], \quad (6)$$

where initially we assume neutrino in a particular flavour state  $\rho_F(0) = |\Psi_F\rangle\langle\Psi_F|$  which in our case chosen to be electron, i.e.  $|\Psi_F\rangle = |\nu_e\rangle$ . Neutrino oscillation can be quantified in a most natural way by considering an overlap between a flavour state solving Eq. (6) and a state of well-defined flavour  $\rho(0) = |\nu_e\rangle\langle\nu_e|$  initializing oscillation Eq. (6). The corresponding probability

$$P_0 = \text{Tr}[\rho_F(0)\rho_F(t)] \quad (7)$$

serves in the following discussion as natural reference (benchmark) for QSL dynamics which properties are compared. Let us notice that for a neutrino initially in a pure state  $\rho_F(0)$ , the overlap  $P_0$  in Eq. (7) is related to the fidelity of states [13]. The terms  $[\mathcal{H}_{\mathcal{F}}, \cdot]$  and  $L[\cdot]$  are responsible for unitary (Hamiltonian generated) and dissipative (Lindbladian) part of time evolution Eq. (6), respectively. The Hamiltonian  $\mathcal{H}_{\mathcal{F}}$  [19, 26,35,50]:

$$\mathcal{H}_{\mathcal{F}} = H_{\text{kin}} + H_{\text{pot}}, \quad (8)$$

includes the kinetic  $H_{\text{kin}}$  and the potential  $H_{\text{pot}}$  part, the latter resulting from neutrino interaction with normal matter. The Lindbladian  $L$  in Eq. (6) describes decoherence and a discussion of its influence on the QSL is postponed to the next Sections. In the absence of decoherence the time evolution of the neutrino, Eq. (6) reduces to the Schrödinger equation. Operating at an ultrarelativistic limit for the neutrinos (with neutrino's eigenenergies  $E_k = E + \frac{m_k^2}{2E}$  and momentum  $p \sim E$ )—the kinetic part of

the Hamiltonian  $\mathcal{H}_{\mathcal{F}}$  in the flavour basis reads:

$$H_{\text{kin}} = \frac{1}{2E} U_{\text{PMNS}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} U_{\text{PMNS}}^\dagger, \quad (9)$$

where  $\Delta m_{ij}^2 = m_i^2 - m_j^2$  is the difference between the masses of two oscillating neutrinos. In the following numerical calculations, we set  $\Delta m_{12}^2 = 7.37 \cdot 10^{-5} \text{ eV}^2$  and  $\Delta m_{31}^2 = 2.52 \cdot 10^{-3} \text{ eV}^2$  [23]. The main contribution to the potential part  $H_{\text{pot}}$  originates from neutrino's interaction due to coherent forward elastic scattering (and for negligible non-coherent effects) with the matter's electrons or neutrons generating *charged-current potential*  $V_C$ . The potential part of the Hamiltonian takes then the following form:

$$H_{\text{pot}} = \begin{pmatrix} V_C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (10)$$

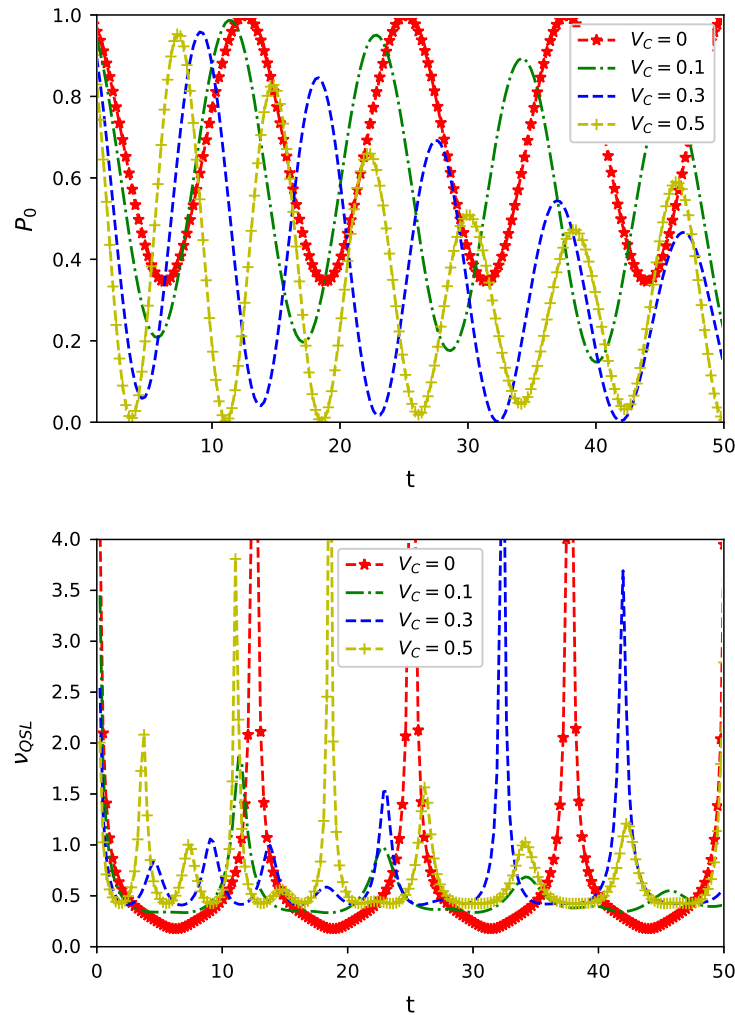
The charged-current potential  $V_C$  is related to the electron's density in the matter  $V_C = \sqrt{2} G_F n_e$  with the Fermi coupling constant  $G_F$  and the electron number density  $n_e$ .

### 3 Results

Quantum speed limit of neutrino oscillation is analysed by means of numerical solution of Eq. (6) obtained using QuTip [38,39], a Python-based toolbox dedicated to dynamics and control of open quantum systems. We present results in three steps: first including normal matter, the second including decoherence effects and finally presenting our central results concerning QSL for  $CP$ -violating system. We investigate numerical solution of Eq. (6) with Hamiltonian and Lindbladian parts given in Eq. (8) and (introduced below) Eq. (11), respectively. We utilize QuTiP master equation solver `qutip.mesolve` (both for unitary and dissipative evolution) for a neutrino initially in the electron flavour. In calculation of quantum speed limit Eq. (2), we utilize `qutip.bures_angle` for Bures distance and `scipy.linalg.svdvals` to compute singular values of  $\mathbb{L}[\rho_F(t)]$  in Eq. (2).

#### 3.1 Normal matter

The charged-current potential  $V_C$  is the first quantity which significantly affects neutrino oscillation modifying its period as presented in an upper panel of Fig. 1. The modification is present also in the geometric QSL as presented in lower panel of Fig. 1. Let us notice that in an absence of normal matter i.e. for  $V_C = 0$  maximal values  $P_0 = 1$  correspond to *discontinuities* of  $\nu_{\text{QSL}} \rightarrow \infty$  what is in agreement with interpretation of geometric QSL as a speed of changes of statistical distance between states [28,29]. Due to the discontinuities (vertical asymptotes) a range of  $\nu_{\text{QSL}}$  for  $V_C = 0$  presented in Fig. 1 is truncated. Increase in  $V_C$  results in changes of periodicity of oscillation and,



**Fig. 1** Upper panel: decoherence-free neutrino oscillation qualified by  $P_0$  Eq. (7) as a function of time for various values of the charge-current potential  $V_C$ . Lower panel: geometric quantum speed limit of decoherence-free neutrino oscillation  $\nu_{QSL}$  Eq. (2) (for clarity of presentation truncated to  $\nu_{QSL} < 4$ ) as a function of time for various values of the charge-current potential  $V_C$ . The  $CP$  violating phase  $\delta = 0$

as a consequence of interaction with normal matter, discontinuities in  $\nu_{QSL}$  disappear cf. Fig. 1 replaced by sharp but continuous peaks. As in the absence of decoherence quantum finite-dimensional systems are periodic or quasi-periodic, one obtains large values of  $\nu_{QSL}$  provided that  $P_0 \approx 1$ , i.e. the time-evolving systems returns in a neighbourhood of its starting point. Inclusion of interaction with matter allows to abandon an artefact of discontinuous QSL time characteristics clearly absent in real systems.

### 3.2 Decoherence

One of peculiarities of neutrino as a fundamental particle is its extremely weak interaction with almost everything existing in the Universe. However, omnipresent decoherence is unavoidable and can serve as one of possible mechanisms [8,9,11,24,31,41,47] affecting unusual properties of a neutrino. We consider the simplest Markovian model

of decoherence based on natural requirement of complete positivity of time evolution [4] leading to a Master equation Eq. (6) with a (non-Hamiltonian) Lindbladian part

$$L[\rho_F] = \sum_{i,j=0}^{N^2-1} c_{ij} \left( F_i \rho_F F_j^\dagger - \frac{1}{2} \{ F_j^\dagger F_i, \rho_F \} \right) \quad (11)$$

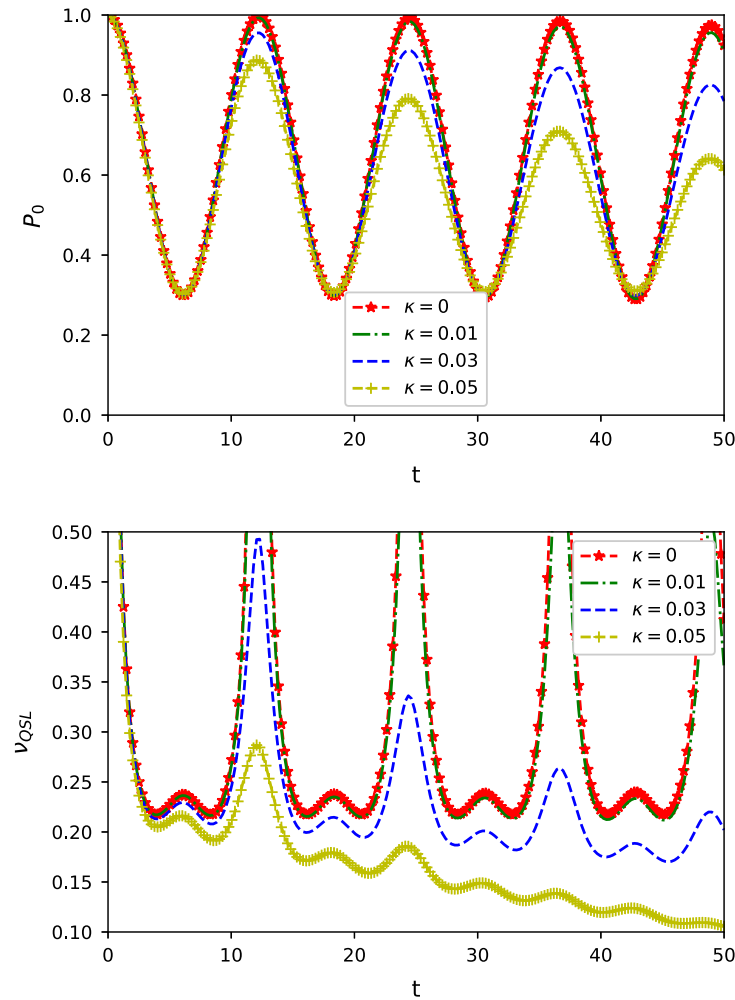
being responsible for the non-standard effects connected with dissipation and decoherence. Here,  $N = 3$  denotes the dimension of the system,  $\{\cdot, \cdot\}$  is an anti-commutator, the matrices  $F_n$  stand for the generators of  $SU(3)$ . In the standard representation, they are given by the celebrated Gell–Mann matrices  $F_n = \frac{\lambda_n}{2}$ , where  $\lambda_n$  [37]:

$$\begin{aligned} \lambda_0 &= \mathbb{I}_3, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \quad (12)$$

The coefficients  $c_{ij}$  in Eq. (11) satisfy the set of inequalities granting complete positivity [11,12]):

$$|c_{ij}| \leq \frac{1}{2}(c_{ii} + c_{jj}). \quad (13)$$

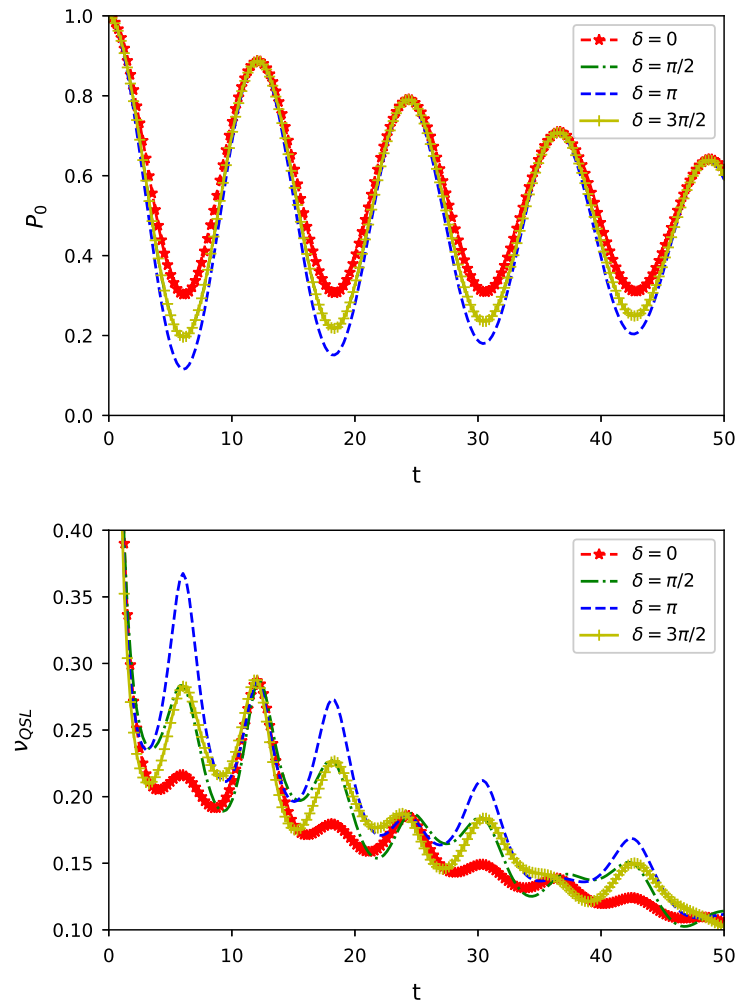
In our analysis, we make one more simplifying step and we limit our consideration to diagonal decoherence with  $c_{ii} = \kappa$  and  $c_{i \neq j} = 0$ . In Fig. 2, we present simultaneous effect of decoherence and very weak interaction with normal matter with  $V_C = 0.03$  in order to avoid artificial discontinuities in  $\nu_{\text{QSL}}$  present in ideal noise-less systems. Let us notice that a range of  $\nu_{\text{QSL}}$  presented in Fig. 2 for a noise-less case,  $\kappa = 0$  is (as irrelevant for further discussion) truncated. Let us also notice initially asymptotic value of  $\nu_{\text{QSL}}$  as  $t \rightarrow 0^+$  due to a definition (and its geometric interpretation) of the geometric speed limit Eq. (2). Markovian decoherence, contrary to previously discussed interaction with normal matter, does not influence periodicity of neutrino oscillation but rather effects in amplitude damping as it can be inferred from an upper panel of Fig. 2. Non-vanishing decoherence removes discontinuities present in a noise-less time evolution of  $\nu_{\text{QSL}}$  but does not affect position of peaks since decoherence of any amplitude  $\kappa$  does not alter positions of extreme values in QSL compared to the noise-less oscillations. For sufficiently large values of  $\kappa$ , there is a qualitative change in the QSL characteristics which, as it is presented in the lower panel of Fig. 2, starts to be monotonically decreasing function of time.



**Fig. 2** Neutrino oscillation quantified by  $P_0$  Eq. (7) (upper panel) and neutrino QSL quantified by  $\nu_{\text{QSL}}$  Eq. (2) for clarity of presentation truncated to  $\nu_{\text{QSL}} < 0.5$  (lower panel) for different amplitudes of decoherence  $c_{ii} = \kappa$  in a presence of normal matter  $V_C = 0.03$  and no  $CP$ -violation  $\delta = 0$

### 3.3 $CP$ -violation

An influence of the  $CP$  symmetry and its potential violation is visible both in decoherence-affected neutrino oscillation and geometric QSL as presented in upper and (respectively) lower panel of Fig. 3. For a better visualization of results, a range of  $\nu_{\text{QSL}}$  is in the lower panel of Fig. 3 truncated. Let us emphasize the presence of initially asymptotic value of  $\nu_{\text{QSL}}$  as  $t \rightarrow 0^+$ . Contrary to previously discussed cases of modification via decoherence only affecting height of  $P_0$  graph, non-vanishing  $CP$  phase  $\delta$  results in increase in  $P_0$  depth which becomes maximal for  $\delta = \pi$ . In other words, neutrino oscillates with the same (quasi)period but in a presence of  $CP$ -violation it wanders off further (in a sense of decreasing fidelity) than it does for  $\delta = 0$ . Resulting behaviour of  $\nu_{\text{QSL}}$  suggests possibility of using geometric QSL to quantify the  $CP$ -violation if one utilizes relation between amplitude of  $\nu_{\text{QSL}}$  and a value of  $\delta$ . However, let us emphasize an existence of time windows where both  $P_0$  and  $\nu_{\text{QSL}}$  become indistinguishable with respect to different  $\delta$  limiting potential



**Fig. 3** Neutrino oscillation quantified by  $P_0$  Eq. (7) (upper panel) and neutrino QSL quantified by  $\nu_{\text{QSL}}$  Eq. (2) for clarity of presentation truncated to  $\nu_{\text{QSL}} < 0.4$  (lower panel) for different values of  $CP$ -violating phase  $\delta$  in the presence of decoherence with  $\kappa = 0.03$  in Eq. (11) and in the presence of normal matter  $V_C = 0.03$

applicability of QSL as a hallmark of violation of the  $CP$  symmetry. Let us notice that the effect of  $CP$ -symmetry is even stronger if one considers  $\nu_{\text{QSL}}$  cf. lower panel of Fig. 3. First, the non-vanishing  $\delta$  enhances peaks of which otherwise are damped by decoherence. As a result, geometric QSL quantifies a potential  $CP$  violation provided that we remember the above mentioned time windows where both  $P_0$  and  $\nu_{\text{QSL}}$  for all values of  $\delta$  are identical. It occurs close to local maxima of  $P_0$ , i.e. at time instance when the neutrino returns close to its initial state.

## 4 Discussion

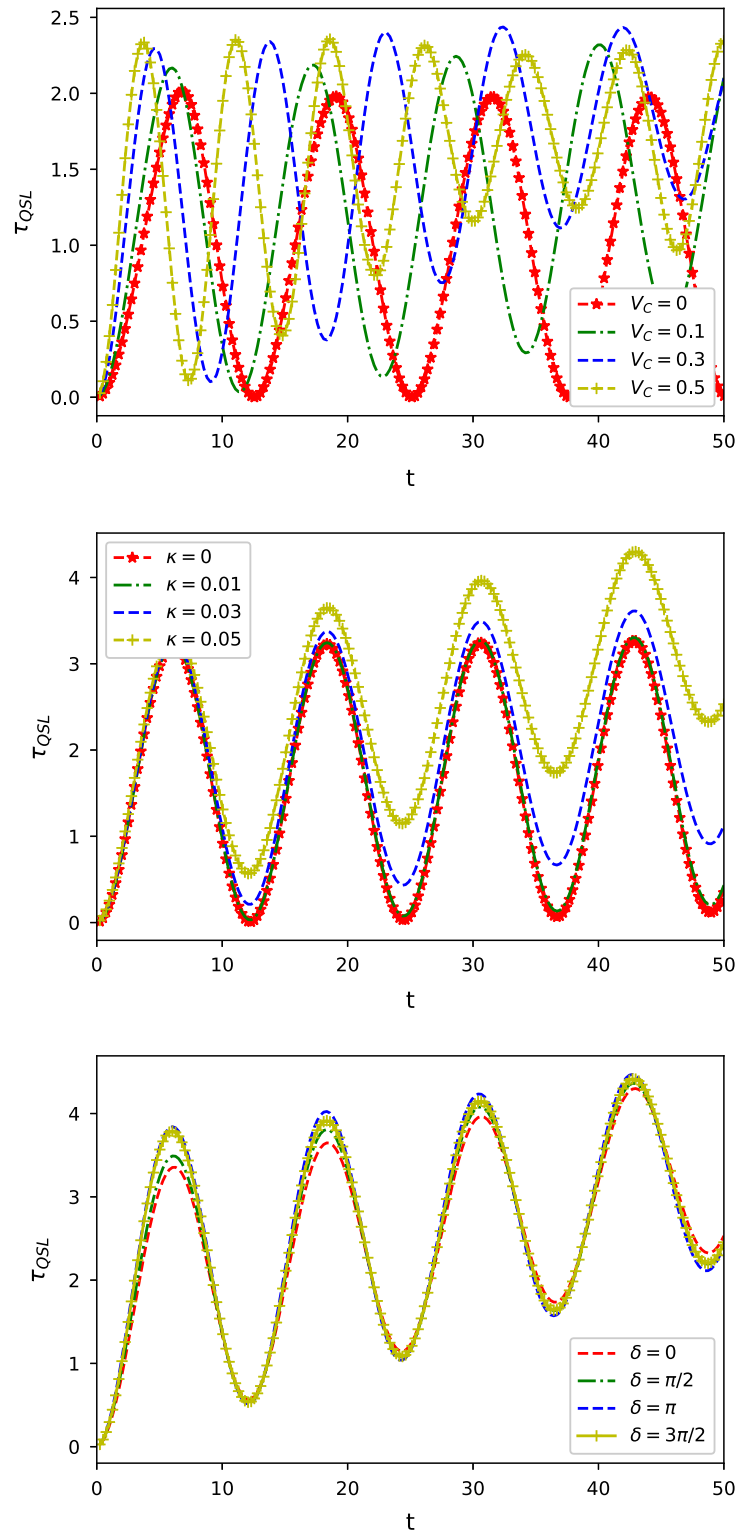
Neutrinos belong to a class of objects allowing for 'complementary' studies. The most common approaches of particle physics community become often success-

fully augmented by results concerning quantum informational properties studied from seemingly different perspective.

Our work belongs to a part of investigations indicating possibility of utilizing quantum informational perspective for better understanding the nature of neutrino. We considered geometric quantum speed limit and we showed its potential applicability as a hallmark for the  $CP$ -violation. First, we presented how QSL as a function of time becomes modified by interaction of neutrino with normal matter formed by matter's electrons or neutrons and generating the charge-current correction  $V_C$  in the Hamiltonian Eq. (8). Removal of artificial discontinuities in the QSL-time characteristics is the main effect of this interaction. Infinite peaks in  $\nu_{\text{QSL}}$  typical for an ideal decoherence-free system become shifted and lowered by  $V_C$  and, for sufficiently large values of  $V_C$  lose their direct connection to the position of extremal values of  $P_0$  Eq. (7). These results after inclusion of decoherence effect become supplemented by damping of oscillations of  $\nu_{\text{QSL}}$  which, for sufficiently high decoherence (given by an amplitude  $\kappa$  of Lindblad dissipators Eq. (11)). Although we used purely phenomenological model of decoherence guided solely by the requirement of complete positivity, the results are generic for Markovian open quantum systems.

The most significant results allow to relate geometric QSL to a very hot topic of hypothetical violation of the  $CP$  symmetry for neutrinos [54]. We infer that such a violation indicated by non-vanishing  $\delta$  in the Pontecorvo–Maki–Nakagawa–Sakata mixing matrix Eq. (5) results in an enhancement of  $\nu_{\text{QSL}}$  amplitude and effective separation of time characteristics of geometric QSL of systems with different  $\delta$ . We also identified time windows of geometric QSL where this effect becomes either particularly visible or vanishes making time characteristics almost  $\delta$ -independent.

Geometric quantum speed limit Eq. (2) quantifies informational (statistical) content of a time-local (calculated at a given time instant  $t$ ) rate of change of neutrino state. There are three factors affecting this rate which are studied in this paper: interaction between neutrino and normal matter (for  $V_C \neq 0$ ), Markovian decoherence (for  $\kappa \neq 0$  in Eq. (11) and potential violation of the  $CP$  symmetry (for  $\delta \neq 0$  in Eq. (5)). From a mathematical perspective, the changes in the geometric speed limit Eq. (2) reported here are due to a collective effect of collaboration of these three factors. In particular, the effect of  $\delta$  in  $\nu_{\text{QSL}}$  starts to be visible provided that one allows for the presence of decoherence in a dynamical model of neutrino oscillation. It is due to an effective cancellation of the  $CP$ -violating phase both for the operator norm and the Bures distance  $l(t)$  (related to fidelity and for pure states given by their overlap) in the numerator and denominator of Eq. (2), respectively, for conservative model of neutrino oscillation with  $\kappa = 0$ . The numerator of QSL Eq. (2) is given by maximal singular value which, to be calculated, demands multiplication of an operator and its conjugate [28] resulting in the above-mentioned cancellation of phase. In the presence of decoherence resulting in an information loss due to non-unitary dynamics Eq. (11), the effect of  $\delta$  is present indicating limiting applicability of purely Hamiltonian models of a description of at least certain properties of neutrino oscillation in time. The QSL quantifying instantaneous rate of changes of neutrino state under non-unitary evolution is a quantity which allows to exhibit otherwise 'invisible' effects related to characteristic time scale of neutrino dynamics. The phenomenon of neutrino oscillations in time and for decoherence-free Hamiltonian description is related



**Fig. 4** Quantum speed limit time  $\tau_{QSL}$  Eq. (14) as a function of time: (i) upper panel: for various values of the charge-current potential  $V_C$  in the absence of decoherence  $\kappa = 0$  and with  $CP$  violating phase  $\delta = 0$ . (ii) Central panel: for different amplitudes of decoherence  $c_{ii} = \kappa$  in the presence of normal matter  $V_C = 0.03$  and no  $CP$ -violation  $\delta = 0$ . (iii) Bottom panel: for different values of  $CP$ -violating phase  $\delta$  in the presence of decoherence with  $\kappa = 0.03$  in Eq. (11) and in the presence of normal matter  $V_C = 0.03$

to the Mandelstam–Tamm time-energy uncertainty relation [16]. Despite of the presence of certain controversial issues [3,15], it allows to set a characteristic time interval required for a significant change of the flavour neutrino state, i.e. an intrinsic time scale of the inter-flavour passage [16]. Let us emphasize that in particular the Mössbauer neutrinos produced in two-body decays of nuclei embedded in a crystal lattice [2] can be utilized for justification and verification of this property [17]. Working beyond conservative dynamics approximation, as we do here, one can utilize QSL as a natural generalization of a (local in time) bound for a rate of neutrino oscillation. Moreover, the geometric quantum speed limit Eq. (2) is directly associated with a quantum speed limit time [28]

$$\tau_{\text{QSL}}^{-1}(t) = \frac{1}{t} \int_0^t v_{\text{QSL}}(t') dt' \quad (14)$$

which thereafter can serve as a generalization of the characteristic time interval for a significant change of the flavour neutrino state in a very general, possibly non-Hamiltonian, systems. The quantum speed limit time Eq. (14) is directly related to geometric quantum speed limit QSL Eq. (2) and, by extension, it reflects the results reported in this work. The quantum speed limit time  $\tau_{\text{QSL}}$  is presented in Fig. 4 for the same set of parameters as previously  $v_{\text{QSL}}$ . Let notice that although the results are qualitative only, one recognizes characteristic features of geometric speed limit Eq. (2) also present in the speed limit time Eq. (14). Following the interpretation given in Refs. [16,17] of the time-energy uncertainty and a related time scale as being characteristic for duration of a neutrino inter-flavour passage, one observes both an interaction with normal matter of an amplitude  $V_C$  and Markovian decoherence quantified by  $\kappa$  in the model Eq. (11) significantly modifying the quantum speed limit time Eq. (14). In particular, the effect of  $V_C$  (as presented in the upper panel of Fig. 4 influences both a magnitude of  $\tau_{\text{QSL}}$  and its periodicity as it was for geometric speed limit Eq. (2) in Fig. 1. Similar modification yet applied to an amplitude only is shared by geometric speed limit and its corresponding speed limit time under Markovian decoherence cf. Fig. 2 and the central panel of Fig. 4, respectively. The effect of  $CP$  violation, seemingly less 'spectacular', is also present (bottom panel of Fig. 4 indicating an enhancement of the quantum speed limit time due to an increasing value of the  $CP$ -violating phase  $\delta$ ). We strongly emphasize that all the neutrino experiments such as the recent one reported in Ref. [54]) belong to the most sophisticated experiments performed so far and our predictions concerning QSL and the corresponding time are nothing but qualitative signatures of certain properties of neutrino oscillation.

## 5 Conclusions

Quantum information processing models [46] such as circuit model [46], topological models, Zidan's model of quantum computing proposed in Refs. [56,57] or quantum communication protocols [36,45,48] can—in principle—be realized using neutrinos [52]. However, despite recent experiments [52], such implementations, together with models using gravitational waves [1] as resource, still remain in a domain of science

fiction [25] rather than reality. Nevertheless, investigations of fundamental properties of neutrino with an emphasis given on  $CP$ -violation [51] inspire researchers of various branches of science. In our work, we provided one more quantifier, the geometric QSL, of dynamic properties of neutrino oscillations. Formal similarity of dynamical model governing neutrino oscillation and qutrit dynamics, which we utilize, is a starting point of many investigations concerning properties of neutrino conducted from a perspective of quantum information. However, expecting all the predictions of quantum information to be directly applicable or translatable to a language of neutrino physics can be due to their redundancy unjustifiable. Dynamics of neutrino oscillation even simplified to a qutrit model remains confined and limited by our (current) knowledge of neutrino's physics. As we currently know, neutrinos are massive and are affected—via charged-current potential  $V_{CC}$ , cf. Eq. (8)—by normal matter. We also expect that neutrinos undergo decoherence Eq. (11) which (for hardly interacting neutrinos) can safely be assumed Markovian. Moreover, most recent experiments [54] remove at least a part of these limitations due to a possible non-preservation of the  $CP$  symmetry. Extension of the domain of studies to geometric quantum speed limit Eq. (2) of  $CP$ -violating oscillation models in a presence of decoherence and normal matter, allowed to identify regimes where the QSL, being essentially dynamical property, becomes modified by changes of equations of motion Eq. (6) induced by all the three above mentioned factors, i.e. normal matter, decoherence and  $CP$ -violation. We recognized tailored range of time where the effect of modifications becomes most significant not only indicating an opportunity of using QSL as a diagnostic tool for neutrino properties, e.g. as a hallmark of the  $CP$  violation, but also expressing quantum informational content of neutrino oscillation in terms of its geometric quantum speed limit.

## Declaration

**Conflict of interest** The authors declare that they have no conflict of interest.

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## 8 (Article C) Histories of Neutrino Oscillation of Consistency Induced by the Presence of Normal Matter

### 8.1 Introduction

The aim of Article C was to utilize Consistent Histories approach to a Quantum System with known dynamics to show how it may be availed to shed light on properties of system that are difficult to replicate or measure experimentally, as in the case of neutrinos.

For continuity, the system taken for the study was neutrino oscillations in the presence of normal matter. A complete set of Histories was written for these oscillations, and Consistency Condition was applied to these Histories to investigate mass state of neutrinos, and whether this consistency depends on the Dirac or Majorana nature of Neutrinos, or hypothetical CP violation.

Consistent History approach, also called Decoherent Histories, goes beyond the standard of Copenhagen Interpretation [C21-C25]. There are applications of such ideas on attempts to gain insight on past behavior of a quantum particle in an interferometric experiment [C26-C29].

As detailed in chapter five of the dissertation, a Quantum History of length  $n$  of a quantum system is a sequence of quantum events at successive times, with one projector for each time representing the event. There is one essential additional requirement imposed, referred to as the single framework rule [C21,C30,C37]. That is that one can assign probabilities to events if they belong to a single framework, binding it to physical properties that are compatible [C21,C30,C31,C37].

Just like a stochastic process, Quantum reasoning based on consistent histories [C37] uses projective decomposition of identity, PDI, of a framework, as a quantum counterpart of the event algebra, summed up in equation 1 of the article.

$$P^j P^j = \delta_{jk} P^k, \sum_j P^j = \mathbb{I}$$

The operators  $P^k \in PDI$  are projectors mapping quantum states on subspaces of a Hilbert space of the quantum system under analysis. At each

time instant, all the quantum properties of a system correspond to elements of a temporal PDI.

The consistency of such histories allows one to assign probabilities to sequences of suitably defined events for a quantum system beyond that which can be achieved with the Born rule only. With this approach, neither the future nor the past states of the system must be determined by the present state. They are only related by the probabilities approaching values which are zero or one for a deterministic time evolution.

The assignment of probabilities for this stochastic process comes from an extension of the Born Rule. Extending Born rule to multi-time history  $M^k$ , one can assign each history its weight [C21,C22,C30,C31,C37]:

$$W(M^k) = \langle K(M^k), K(M^k) \rangle = \text{Tr}[K^{dagger}(M^k)K(M^k)]$$

where for a time evolution operator  $T(t_j, t_j)$ ,

$$K(M^k) = P^n T(t_n, t_{n-1}) P^{n-1} \dots P^1 T(t_1, t_0) P^0$$

is the Chain Operator [C21].

Assigning probabilities to non-commuting quantum properties is meaningful only if there is no interference between pairs of histories that are supposed to be decoherent, which gives rise to the "Consistency Condition". It comes from the rules of classical probability that elements of a sample space are mutually exclusive [C21], and reads as follows [C21,C22,C30,C31,C37]:

$$\langle K(M^k), K(M^s) \rangle = 0 \text{ for all } k \neq s,$$

which is equation 6 of article C.

Thus, Consistent History Approach requires the designing of complete family of histories, in the sense detailed in chapter 5, highlighting the properties of the system to be studied within these histories, and using the consistency condition to find the parameters of the time evolution of the system that enable the "accessibility" of these properties, especially in the cases where the said property might not be practically accessible.

## 8.2 Model

Article C considered a simplified two-flavor model (qubit) of neutrino oscillation limited to the electron–muon neutrino oscillation [C1,C2,C4,C17,C18].

This simplicity, coupled with dynamics that do not include dissipation, allowed for an analytical solution, but despite the limitations, it still allows one to understand some of the origins and mechanisms behind the oscillations.

Details of qubit approximation of neutrino oscillation were discussed in the second chapter of the dissertation. There are two orthogonal vacuum-massive states  $|1\rangle$  and  $|2\rangle$ , and two flavor states  $|v_e\rangle$  and  $|v_\mu\rangle$  that were considered.

There are four parameters that enter the total Hamiltonian of a qubit neutrino propagating in matter. One mixing angle,  $\theta_{12}$ , difference between the two masses  $\Delta m_{21}$ , Majorana phase (CP-violating)  $\phi$ , and the interaction potential with normal matter due to coherent forward scattering on electrons via charged current  $V_0$ . The total Hamiltonian, referred to as the Neutrino Wolfenstein effective Hamiltonian [C1,C4], was presented in the second chapter.

A Majorana type neutrino was considered for this article to determine if consistency of a family of histories of neutrino oscillations can have a say on the Dirac or Majorana nature of neutrino.  $\phi$  is non-zero under the  $U(1)$  global transformation in the Majorana case, unlike that in the Dirac case, where this transformation eliminates it [C1].

The shortest nontrivial three-time histories of the two-flavor neutrino evolution was designed in the article. The beginning and ending points of the histories are "events" of a neutrino in a flavor state. However, as the intermediate event of the three-time history, there is a neutrino in a pure mass state, equation 11 of the article.

$$M^0 : [v_\mu] \odot [I] \odot [I]$$

$$M^1 : [v_e] \odot [1] \odot [v_\mu]$$

$$M^2 : [v_e] \odot [1] \odot [v_e]$$

$$M^3 : [v_e] \odot [2] \odot [v_\mu]$$

$$M^4 : [v_e] \odot [2] \odot [v_e],$$

where  $[x] = |x\rangle\langle x|$ , is the projector for finding neutrino is state  $|x\rangle$ . Mathematically, the sequences of events are described as the tensor product of 'instantaneous' Hilbert spaces. The study applied  $\odot$  instead of  $\otimes$ , as a symbol for temporal (multi-time), rather than spatial (multi-partite) composites.

The objective was to ascertain whether the neutrino being in its mass state is an event of a consistent ontological value [C31]. This family of histories describes different possibilities (histories) starting from an electron neutrino only, but an analogous family starting with a muon neutrino can be defined and all the results presented below remain valid for such a choice.

### 8.3 Results

The chain operators for each history made up for equation 12 of the article:

$$\begin{aligned}
K(M^0) &= T_2 T_1 | v_\mu \rangle \langle v_\mu | \\
K(M^1) &= \langle v_\mu | T_2 | 1 \rangle \langle 1 | T_1 | v_e \rangle \cdot | v_\mu \rangle \langle v_e | \\
K(M^2) &= \langle v_e | T_2 | 1 \rangle \langle 1 | T_1 | v_e \rangle \cdot | v_e \rangle \langle v_e | \\
K(M^3) &= \langle v_\mu | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \cdot | v_\mu \rangle \langle v_e | \\
K(M^4) &= \langle v_e | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \cdot | v_e \rangle \langle v_e |
\end{aligned}$$

$T_1$  and  $T_2$  represent the time development operators for the first and second evolution the system. Consistency is achieved if  $\langle K(M^i) | K(M^j) \rangle = 0$  for all  $i \neq j$ .

All histories are mutually exclusive to each other regardless of the dynamics of the system, apart from  $M^1$  and  $M^3$ , and  $M^2$  and  $M^4$ . Equation 13 expanded the overlap between these histories as:

$$\begin{aligned}
\langle K(M^1) | K(M^3) \rangle &= \langle 1 | T_2^\dagger | v_\mu \rangle \langle v_e | T_1^\dagger | 1 \rangle \langle v_\mu | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \\
\langle K(M^2) | K(M^4) \rangle &= \langle 1 | T_2^\dagger | v_e \rangle \langle v_e | T_1^\dagger | 1 \rangle \langle v_e | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle
\end{aligned}$$

Consistent histories approach inherently depends on the probability distribution used to assign probabilities of quantum events, and, thus, on the dynamics of the system [C21]. Thus, solving for consistence for this family of events is to find conditions or parameters that satisfy the below:

$$\begin{aligned}
\langle K(M^1) | K(M^3) \rangle &= 0 \\
\langle K(M^2) | K(M^4) \rangle &= 0
\end{aligned}$$

The article first tried to achieve consistency for a neutrino propagating in vacuum. As  $V_0 = 0$  for this case, the effective Hamiltonian is:

$$H = H_0 = \begin{pmatrix} E - \frac{\Delta m_{21}^2}{4E} & 0 \\ 0 & E + \frac{\Delta m_{21}^2}{4E} \end{pmatrix}.$$

Scaling the initial time to 0, solving for consistency in case of vacuum led to a set of three equations, equation 15 of the article, with at least one of them needing to be satisfied:

$$\begin{aligned} \cos\left(t\left(E - \frac{\Delta m_{21}^2}{4E}\right)\right) - i\sin\left(t\left(E - \frac{\Delta m_{21}^2}{4E}\right)\right) &= 0 \\ \cos\left(t\left(E + \frac{\Delta m_{21}^2}{4E} + \frac{\phi}{t}\right)\right) - i\sin\left(t\left(E + \frac{\Delta m_{21}^2}{4E} + \frac{\phi}{t}\right)\right) &= 0 \\ \cos\left(t\left(E + \frac{\Delta m_{21}^2}{4E} - \frac{\phi}{t}\right)\right) - i\sin\left(t\left(E + \frac{\Delta m_{21}^2}{4E} - \frac{\phi}{t}\right)\right) &= 0 \end{aligned}$$

Either the real parts of each equation could be zeroed, or the imaginary parts, but not both simultaneously. Thus, it was concluded that in the absence of matter, the family of histories is never consistent under the dynamics generated by  $H_0$ , and one cannot credibly assign probability to the events in this family of histories.

The article then showed how this changes once the neutrino propagates in the presence of normal matter. The effective Hamiltonian for a two-level neutrino propagating in normal matter that was presented in chapter two, looks like equation 16 of the article when diagonalized.

$$H = \frac{1}{2}(A + B + D) | X \rangle \langle X | + \frac{1}{2}(A + B - D) | Y \rangle \langle Y |$$

where

$$\begin{aligned} A &= E - \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 + \cos 2\theta) \\ B &= E + \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 - \cos 2\theta) \\ C &= \frac{V_0}{2} \sin 2\theta \\ D &= \sqrt{(A - B)^2 + 4C^2} \end{aligned}$$

The Eigenstates are given as below:

$$\begin{aligned}|X\rangle &= \frac{1}{u} \left( |1\rangle + \frac{1}{2} \frac{e^{i\phi}(B-A+D)}{C} |2\rangle \right) \\ |Y\rangle &= \frac{1}{v} \left( |1\rangle + \frac{1}{2} \frac{e^{i\phi}(B-A-D)}{C} |2\rangle \right),\end{aligned}$$

with

$$\begin{aligned}u &= \frac{\sqrt{4C^2 + e^{i2\phi}(B-A+D)^2}}{2C}, \\ v &= \frac{\sqrt{4C^2 + e^{i2\phi}(B-A-D)^2}}{2C}.\end{aligned}$$

For further convenience, article C also formulated the below:

$$\begin{aligned}J &= \cos\theta(A-B+D) + 2\sin\theta C, \\ K &= \cos\theta(A-B-D) + 2\sin\theta C, \\ L &= 2\cos\theta C - \sin\theta(A-B+D), \\ N &= \sin\theta(A-B-D) - 2\cos\theta C.\end{aligned}$$

Utilising the above, flavor states made up equation 19 of the article:

$$\begin{aligned}|v_e\rangle &= \frac{1}{2D} \left[ uJ |X\rangle - vK |Y\rangle \right], \\ |v_\mu\rangle &= \frac{1}{2D} \left[ uL |X\rangle + vN |Y\rangle \right].\end{aligned}$$

Just like the consistency check for vacuum scenario, a set of equations is obtained for neutrino oscillations in normal matter, and the article shows that at least one of the 4 equations needs to be satisfied to obtain consistency.

$$\begin{aligned}|u|^2 L(A-B+D) - |v|^2 N(A-B-D)e^{itD} &= 0 \\ |u|^2 J(A-B+D) + |v|^2 K(A-B-D)e^{itD} &= 0 \\ |u|^2 L - |v|^2 Ne^{itD} &= 0 \\ |u|^2 J + |v|^2 Ke^{itD} &= 0\end{aligned}$$

And just like the vacuum scenario, the equations are split into their corresponding real and imaginary parts, coming from the nontrivial imaginary

parts contained in  $u$  and  $v$ . Rest of the parameters are real. The imaginary parts are presented in equation 22 of the article.

$$|v|^2 N(A - B - D) \sin(tD) = 0$$

$$|v|^2 K(A - B - D) \sin(tD) = 0$$

$$|v|^2 N \sin(tD) = 0$$

$$|v|^2 K \sin(tD) = 0$$

Unlike the case of propagation in vacuum, the article reasoned and showed that in the presence of vacuum, the imaginary parts of each equation zero when  $tD = n\pi$ ,  $n \in [N]$ , and only so as any other possibilities were shown to be not possible.

The article then further calculated the parameters that would result in the real parts being simultaneously satisfied when the time between events satisfied  $tD = n\pi$ . The corresponding real parts for each equation were simplified in equation 24 of the article, and at least one needed to be satisfied for consistency.

$$(B - A + D)L(A - B + D) + (B - A - D)N(A - B - D)\cos(tD) = 0$$

$$(B - A + D)J(A - B + D) - (B - A - D)K(A - B - D)\cos(tD) = 0$$

$$(B - A + D)L + (B - A - D)N\cos(tD) = 0$$

$$(B - A + D)J - (B - A - D)K\cos(tD) = 0$$

As  $\cos(tD)$  enters each equation, and  $tD = n\pi$ , there were two cases to consider,  $n$  being an even number or an odd number. For  $n$  being even, each equation was satisfied for  $D = 0$ , which was rejected as non-physical as it would imply both eigenenergies being equal and contradict the neutrino oscillations.

As a reminder, at least one from the set of equations needed to be satisfied. For  $n$  odd, the non-negative solution to two of the equations was obtained for  $V_0 = -\frac{\Delta m_{21}^2}{2E}$ . The article showed that this family of histories, where a neutrino is in a pure mass state as an intermediate event between two pure flavor states, is consistent in the presence of normal matter, when  $V_0 = \frac{\Delta m_{21}^2}{2E}$ ,  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E}$ , and  $n$ , which is equal to  $\frac{tD}{\pi}$ , is an odd number.

## 8.4 Conclusion

Chapter five discussed how histories of vanishing probabilities and inconsistent families of histories are distinct from each other. Histories with vanishing probability  $K(M) = 0$  never occur and are termed as dynamically impossible. Using time development governed by the laws of quantum dynamics, under the Consistent Histories Approach, inconsistent family of histories cannot be assigned any probabilities, and one must refrain from the temptation of thinking of an inconsistent family of histories as one that never occurs (probability equals 0) [C21,C31].

There are two intervals between the three events in each history. Whether  $t_1 = t_2$ , and the choice of  $D$  significantly affects probabilities (weights) of the histories. If  $t_1 \neq t_2$  and  $n$  is an odd number such that we have  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t_1}$ , and  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , the family is consistent. However,  $M_1$  and  $M_2$  then have zero weights, and these events do not occur at all.

If  $t_1 \neq t_2$  and  $n$  is an odd number such that we have  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t_2}$ , and  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , the family is also consistent. However,  $M_2$  and  $M_3$  then have zero weights.

When  $t_1 = t_2 = t$ ,  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t}$ , and  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , such that  $n$  is an odd number, the family of histories is consistent. However,  $M_1$ ,  $M_2$  and  $M_3$  then have zero weights.

In Conclusion, the consistent histories approach allows one to describe quantum time development as a process that is essentially and always stochastic or probabilistic, and not just when measurements are being carried out. The consistency condition allows one to decide which among the potential histories can occur or can even be considered. The article investigated if a class of simple, three-time histories (including quantum property of a neutrino having a definite mass) can be consistent.

The family of events proposed was inconsistent when a neutrino propagated in vacuum. However, under certain tailored dynamical conditions, the family of histories became consistent when a neutrino propagated in the presence of normal matter, provided that an interaction potential,  $V_0$ , and the time interval(s) between the events took particularly exact values.

Confronting this with some known experimental values gives further insight.

As per [C38], estimates for solar neutrinos put  $\Delta m_{21}^2 = 1.4906 \times 10^{-60} m_P$ ,  $(8 \times 10^{-5} eV^2)$ . The magnitudes of  $V_0$  and  $D$  (which represents the difference between the two eigenvalues of the system) that provide consistency are extremely small for even a relatively meager value of  $E$ . Thus, the article provided additional evidence for the neutrino mass states being hardly accessible, even from a fundamental perspective, and not only a technical viewpoint.

The consistency of the class of histories investigated in the article was not affected by the CP-violating Majorana phase. However, the results presented hold true for both Majorana or Dirac neutrinos and would be unchanged after the CP problem is also resolved.

## Article

# Histories of Neutrino Oscillation of Consistency Induced by the Presence of Normal Matter

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**Abstract:** Oscillation of two-flavour neutrinos is considered within a quantum mechanical framework of consistent (decoherent) dynamic histories. We investigate how consistency of selected three-time histories is affected by oscillation parameters. We show that the presence of normal matter is crucial to maintain consistency of certain classes of neutrinos' dynamic histories and that the consistency does not depend on a Majorana phase and remains insensitive to a potential CP violation.

**Keywords:** neutrino oscillation; consistent histories; CP violation



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## 1. Introduction

A measurement allows gaining knowledge about the present of quantum systems. It is, however, difficult to infer from this what the system's past had been. It is even harder to say something about quantum systems that are essentially closed (such as, for example, the whole universe is) and which do not allow for an external observer, or those that are simply resistant to measuring attempts. Neutrinos, hardly interacting with anything, can serve as a natural example of such a 'measurement-resistant' system. Time-evolving neutrinos [1,2] and their oscillations are an example of quantum systems that are particularly difficult to measure. Despite tremendous progress in experimental techniques, even the latest and most sophisticated experiments [3] do not resolve all doubts concerning the fundamental properties of these mysterious and fascinating particles, stimulating research from various perspectives. The dynamic properties of neutrino oscillations [4–6], initially studied in the most natural particle physics [1] context, are rich of various aspects typically related to quantum information processing [5,7–13]. Such a broad and multi-perspective interest is motivated not only by the natural applicability of methods and computational techniques borrowed from quantum information processing in the domain of particle physics, but also by recent proposals of information transfer protocols utilizing neutrinos [14] as a resource, or even reflecting the enduring human dream of interstellar communication [15].

The discovery of neutrino oscillations was the first evidence in favour of physics beyond the Standard Model, obviously attracting the particle community [16]. Complementary studies of neutrino oscillations from the perspective of quantum information and communication are naturally supported by an effective description of neutrinos, after limiting to a simplified two-flavour model, as a qubit [4,10,12,13,17–19]. The time evolution of a neutrino qubit originates from a fundamental non-correspondence between the neutrino's flavour  $\{|\nu_e\rangle, |\nu_\mu\rangle\}$  and massive states  $|1\rangle, |2\rangle$ . Motivation for, and even the importance of, such research is additionally boosted by results of recent neutrino experiments [3] regarding possible non-trivial CP-violations. Potentially violated CP symmetry may indicate a need for the reformulation of many fundamental aspects of the universe from the micro to macro scales. It is because of the charge-conjugation parity reversal that CP symmetry between

matter and antimatter can be questioned in light of the results of the above-mentioned experiment [3].

Making a measurement and finding a quantum particle in a particular quantum state does not allow inferring its state in the past. Orthodox researchers claim that one cannot talk about a quantum particle between measurements at all. This obvious limitation of standard quantum theory inspires its extensions. The two-state vector formalism [20] and consistent (or decoherent) histories approaches [21–25] are but two fruitful examples going beyond (and sometimes across) the Copenhagen interpretation. We have recently been witnessing how the two above-mentioned approaches are competitively applied to a problem of the past behaviour of a quantum particle in an interferometric experiment [26–29]. Consistent histories, which we further utilize in this paper, are known to gain an additional insight to problems ranging from the small—but fundamental [30–33]—to the largest of scales [34,35]. The consistency of histories allows one to assign probabilities to sequences of suitably defined events for a quantum system beyond that which can be achieved with the Born rule only. As the quantum events in this perspective do not rely on the notion of measurement per se, the consistent histories approach enables one to design a new type of logical approach [22]. This approach is essentially different than the Birkhoff and von Neumann quantum logic [36]. There is a particular practical advantage of using consistent histories formalism to gain information about a system if an external measurement is not available either because of fundamental reasons, or simply technical reasons. Quantum reasoning based on consistent histories [37] uses projective decomposition of identity  $PDI = \{P^k\}$ , a framework,

$$P^j P^k = \delta_{jk} P^k, \quad \sum_j P^j = I \quad (1)$$

as a cornerstone [21], serving as a quantum counterpart of the event algebra used in standard stochastics. The operators  $P_k \in PDI$  are projectors mapping quantum states on subspaces of a Hilbert space of the quantum system under analysis. If the subspace is one dimensional,  $P_k = |k\rangle\langle k|$  where  $|k\rangle$  is a vector spanning the subspace. Projectors  $P_k \in PDI$  correspond to quantum properties i.e., features of a quantum system. Contrary to standard quantum logic [36], in the consistent histories approach, logical operations, such as  $k_1 \wedge k_2$ , can be applied to the quantum properties  $k_{1,2}$  provided that the properties are compatible, i.e., they correspond to commutative projectors  $[P_{k_1}, P_{k_2}] = 0$  with an obvious relation to the co-measurability of the observables in ‘usual’ quantum mechanics.  $PDI$ s equipped with the admissible logical operation are (for the consistent histories approach) natural analogues of event algebras in standard stochastics.

A quantum history (an analogue of a random process) of length  $n$  of a quantum system is a sequence of quantum events at successive times  $\{P_i\}$ , with one projector for each time  $t_i, i = 0, \dots, f$ . There is, however, one fundamental additional requirement imposed: it is the *single framework rule* [21,30,37] bounding the meaningfulness of simultaneous reasoning to physical properties that are compatible [21,30,31,37]. In other words, one cannot formulate meaningful statements concerning the properties of observables unless they are co-measurable. In particular, one can assign probabilities to events provided that they belong to a single framework. This requirement imposed by the consistent histories approach allows one to dissolve otherwise paradoxical features [30,37]. At each time instant,  $t_i$ , all the quantum properties of a system correspond to elements of a  $PDI(t_i)$ . The properties have their assigned probabilities and a time evolution of the quantum system studied with consistent histories model can be considered as a stochastic process. With such an approach, neither the future nor the past states of the system must be determined by the present state. Instead, they are (only) related by the probabilities approaching particular

values which are zero or one for a deterministic time evolution. A mathematical way to describe sequences of events is the tensor product of ‘instantaneous’ Hilbert spaces  $\mathcal{H}(t_i)$

$$\check{\mathcal{H}} = \bigodot_{i=0}^n \mathcal{H}(t_i) \quad (2)$$

where, following [21], we applied  $\odot$  instead of  $\otimes$  as a symbol for temporal (multi-time) rather than spatial (multi-partite) composites. Let us note, however, that in most cases (and, in particular, further in this work) the time dependence of a system’s Hilbert space is trivial i.e.,  $\mathcal{H}(t_i) = \mathcal{H}$  for  $i = 0, \dots, n$ . In formal terms, quantum properties (events) related to time-evolving systems are its histories forming time-dependent *PDI*

$$M^j M^k = \delta_{jk} M^k, \text{ where } \sum_j M^j = \bigodot_{i=0}^n I = \check{I} \in \mathcal{B}(\check{\mathcal{H}}) \quad (3)$$

$$M^k = \bigodot_{i=0}^n P^{(i,k)}, \quad P^{(i,k)} \in PDI(t_i)$$

where (generically)  $PDI(t_i) \neq PDI(t_j)$  for  $j \neq i$ . Then, there is a chance to pose different questions concerning different quantum properties of the systems at different time instants. Assigning probabilities to non-commuting quantum properties in Equation (3) is meaningful only if there is no interference between pairs of histories that are supposed to be decoherent. After extending the celebrated Born rule to multi-time history  $M^i$  (the consistent histories formalism exhibits, here, its true predictive power), one can assign it a weight [21,22,30,31,37]:

$$W(M^k) = \langle K(M^k), K(M^k) \rangle = \text{Tr} [K^\dagger(M^k) K(M^k)] \quad (4)$$

where for a time evolution operator  $T(t_i, t_j)$

$$K(M^k) = P^{(n,k)} \prod_{i=0}^{n-1} T(t_{i+1}, t_i) P^{(i,k)} = P_n T(t_n, t_{n_1}) P_{n-1} \cdots P_1 T(t_1, t_0) P_0 \quad (5)$$

is termed as the chain operator [21]. Then the *consistency condition* [21,22,30,31,37] granting the decoherence of histories, reads as follows:

$$\langle K(M^k), K(M^s) \rangle = 0, \text{ for all } k \neq s \quad (6)$$

The rationale behind Equation (6) [21] is rooted in standard probability theory. As the histories Equation (3) form a *PDI*, the meaningful linear combinations  $M = \sum_i c_i M^i$  can also be constructed. The corresponding weight reads as follows:

$$W(M) = \sum_i c_i W(M^i) \quad (7)$$

On the other hand, the linearity of  $K(M) = \sum_i c_i K(M^i)$  Equation (4) implies

$$W(M) = \sum_i \sum_j c_i c_j \langle K(M^i), K(M^j) \rangle \quad (8)$$

Let us note that Equations (7) and (8) can be satisfied provided that Equation (6) is granted.

Let us note that a quantum history of a physical system is a sequence of quantum events at successive times, where a quantum event at a particular time can be any quantum property of the system in question [21]. Such a convenient tool can serve to analyse properties of systems that are very difficult to measure or replicate experimentally, as in

the case of neutrinos. A neutrino in a particular mass state is an example of ‘an event’ of a quantum property that is hard (maybe technically impossible) to measure. It is because the neutrino masses corresponding to different mass states are extremely small, and probably too small to resolve experimentally. That is why one is left with considering coherent superposition of the mass states: the flavour states. As a result, one recognizes two classes of neutrino oscillation experiments. In the first, having a beam of a given flavour, one may investigate how many neutrinos have disappeared. In the second, starting with a pure beam of a particular and known flavour, one may investigate how often neutrinos of a different flavour are detected [2].

In this paper, we consider a simplified two-flavour model of neutrino oscillation limited to the electron–muon neutrino oscillation [1,2,4,17,18]. Despite the simplicity of the two-flavour model (as it is analytically treatable), it allows one to understand some of the origins and mechanisms behind the oscillations. We also consider the shortest non-trivial three-time histories of the two-flavour neutrino evolution. Such histories cannot be analysed solely within standard Born rule and the formalism of consistent histories allows one to investigate their properties inaccessible from the standard treatment. The beginning and ending points of the histories are events (in the sense described above) of a neutrino in a flavour state. However, as the intermediate event of the three-time history, there is a neutrino in a *mass state*. According to the decoherent histories model [21], such histories, provided that they are consistent, allow inferring otherwise non-measured, but in principle measurable, quantum properties of the oscillating neutrino. Our primary aim is to investigate consistency of such histories induced by the presence of normal matter affecting the oscillation. We show that modification of neutrino dynamics due to this presence of matter makes the histories otherwise belonging to inconsistent family of dynamical events consistent. We show that a potential CP-violation (indicated by non-vanishing Majorana phase) does not affect the consistency of the considered neutrino histories, and, hence, our findings hold true regardless of neutrino being a Dirac or Majorana particle. We show that the presence of normal matter is a feature that plays a crucial role in the dynamic properties and consistency of the neutrino oscillation considered here.

## 2. Histories of Two-Flavour Neutrino Oscillation

Working in a two-flavour approximation, we consider two orthogonal vacuum-massive states,  $|1\rangle$  and  $|2\rangle$ , and two neutrino flavours: electron ( $e$ ) and muon ( $\mu$ ). The electron neutrino state is given by  $|v_e\rangle = \cos\theta |1\rangle + e^{i\phi} \sin\theta |2\rangle$ , and the muon neutrino state is  $|v_\mu\rangle = -\sin\theta |1\rangle + e^{i\phi} \cos\theta |2\rangle$ , where  $\theta = \theta_{12}$  is the mixing angle. The phase  $\phi$  is the Majorana phase (CP violating), which is non-zero under the  $U(1)$  global transformation in the Majorana case, unlike that in the Dirac case, where this transformation renders it zero, or eliminates it [1]. A vacuum Hamiltonian-generating oscillation of a relativistic neutrino [1] is given by

$$H_0 = \begin{pmatrix} E - \frac{\Delta m_{21}^2}{4E} & 0 \\ 0 & E + \frac{\Delta m_{21}^2}{4E} \end{pmatrix} \quad (9)$$

where  $\Delta m_{21}^2 = m_2^2 - m_1^2$  is the square-mass splitting of the mass states in the normal hierarchy case and  $E$  is the energy of the massless neutrino. The neutrino Wolfenstein effective Hamiltonian [1,4] while propagating in matter is given by

$$H = H_0 + \frac{V_0}{2} \begin{pmatrix} 1 + \cos 2\theta & e^{-i\phi} \sin 2\theta \\ e^{i\phi} \sin 2\theta & 1 - \cos 2\theta \end{pmatrix} \quad (10)$$

where  $V_0 = \sqrt{2} \cos^2 \theta_{13} G_F N_e$ ,  $0.953 \leq \theta_{13} \leq 1$  [1] is one of the oscillation parameters with  $3\sigma$  bound,  $N_e$  is the electron density and  $G_F$  is the Fermi constant.  $V_0$  is the effective interaction potential due to oscillation of the neutrino in the presence of ordinary matter. It results from the coherent forward scattering on electrons via charged current interactions [1].

The corresponding charged current interaction term for muon is not included, as only the usual matter is considered. As per the standard model, the neutral current interaction does not enter effectively into the Hamiltonian [1].

Information gained from applying the consistent history approach to quantum dynamics becomes non-trivial if one considers at least three-time histories. Such histories cannot be directly analyzed with the Born rule [21]. A family of three-times histories of quantum events at  $t_0$ ,  $t_1$  and  $t_2$  that is going to be considered, here, is the following:

$$\begin{aligned} M^0 &: [v_\mu] \odot [I] \odot [I] \\ M^1 &: [v_e] \odot [1] \odot [v_\mu] \\ M^2 &: [v_e] \odot [1] \odot [v_e] \\ M^3 &: [v_e] \odot [2] \odot [v_\mu] \\ M^4 &: [v_e] \odot [2] \odot [v_e] \end{aligned} \quad (11)$$

where  $[x] = |x\rangle\langle x|$ ,  $x = v_{e,\mu}, 1, 2$ . As discussed in the previous section, the presence of  $[x = 1]$  or  $[x = 2]$  as a middle event in the histories corresponds to a mass state as neutrino's quantum property. All the histories in Equation (11), except  $M^0$ , which is included to obtain a temporal *PDI*, describe a neutrino starting in an electron flavour and finishing in one of two flavours: muon or electron. Our objective is to verify if one can consistently include a quantum event  $[i]$ ,  $i = 1, 2$  in the oscillation history i.e., whether the neutrino, being in its mass state, is an event of a consistent ontological value [31]. The family Equation (11) describes different possibilities (histories) starting from an electron neutrino only, but an analogous family starting with a muon neutrino can be defined and all the following results remain valid for such a choice. The chain operators

$$\begin{aligned} K(M^0) &= T_2 T_1 |v_\mu\rangle\langle v_\mu| \\ K(M^1) &= \langle v_\mu | T_2 | 1 \rangle \langle 1 | T_1 | v_e \rangle \cdot |v_\mu\rangle\langle v_e| \\ K(M^2) &= \langle v_e | T_2 | 1 \rangle \langle 1 | T_1 | v_e \rangle \cdot |v_e\rangle\langle v_e| \\ K(M^3) &= \langle v_\mu | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \cdot |v_\mu\rangle\langle v_e| \\ K(M^4) &= \langle v_e | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \cdot |v_e\rangle\langle v_e| \end{aligned} \quad (12)$$

grant consistency of the corresponding histories provided that  $\langle K(M^i) | K(M^j) \rangle = 0$  for all  $i \neq j$ . Applying the consistency condition to all members of the family, we see that all histories are mutually exclusive of each other, regardless of the dynamics of the system apart from  $M^1$  versus  $M^3$ , and  $M^2$  versus  $M^4$ . We have

$$\begin{aligned} \langle K(M^1) | K(M^3) \rangle &= \langle 1 | T_2^\dagger | v_\mu \rangle \langle v_e | T_1^\dagger | 1 \rangle \langle v_\mu | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \\ \langle K(M^2) | K(M^4) \rangle &= \langle 1 | T_2^\dagger | v_e \rangle \langle v_e | T_1^\dagger | 1 \rangle \langle v_e | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \end{aligned} \quad (13)$$

where  $T_1 = \exp[i(t_0 - t_1)H]$  and  $T_2 = \exp[i(t_1 - t_2)H]$  represent the time evolution operators for the first and second interval of the three-time histories Equation (11). The aim is to recognize conditions turning the family of histories Equation (11) consistent:

$$\begin{aligned} \langle K(M^1) | K(M^3) \rangle &\equiv 0 \\ \langle K(M^2) | K(M^4) \rangle &\equiv 0 \end{aligned} \quad (14)$$

### 2.1. Vacuum-Only Oscillation

Consistent histories approach inherently depends on the probability distribution used to assign probabilities of quantum events, and, thus, on the dynamics of the system [21]. For further simplicity, we set initial time  $t = 0$  and we compute the inner products between the chain operators for  $M^1$  and  $M^3$ , and between  $M^2$  and  $M^4$  Equation (13) to equate them

to zero, and obtain the consistency Equation (14). To achieve the consistency Equation (14) in vacuum (i.e., for the time evolution generated by  $H = H_0$  Equation (9)), at least one of the three equations that follow need to be satisfied:

$$\begin{aligned} \cos\left(t\left(E - \frac{\Delta m_{21}^2}{4E}\right)\right) - i \sin\left(t\left(E - \frac{\Delta m_{21}^2}{4E}\right)\right) &= 0 \\ \cos\left(t\left(E + \frac{\Delta m_{21}^2}{4E} + \frac{\phi}{t}\right)\right) - i \sin\left(t\left(E + \frac{\Delta m_{21}^2}{4E} + \frac{\phi}{t}\right)\right) &= 0 \\ \cos\left(t\left(E + \frac{\Delta m_{21}^2}{4E} - \frac{\phi}{t}\right)\right) - i \sin\left(t\left(E + \frac{\Delta m_{21}^2}{4E} - \frac{\phi}{t}\right)\right) &= 0 \end{aligned} \quad (15)$$

where  $t$  is either  $t_1$ , or  $t_2$  representing time intervals between the first and second events in the histories Equation (13). Let us note that either the real parts in Equation (15) can be zeroed, or the imaginary parts, but not both simultaneously. Thus, in the absence of matter, the family of histories Equation (11) is never consistent under the dynamics generated by  $H_0$  in Equation (9). In other words, in the absence of matter, one cannot credibly assign consistency, and, hence, probability, to the events in Equation (11). It changes, however, in the presence of matter.

## 2.2. The Impact of Matter

Let us now consider a neutrino propagating in matter. The effective Hamiltonian describing two-flavour neutrino oscillation in the presence of normal matter Equation (10), written in its diagonal representation, reads as follows:

$$H = \frac{1}{2}(A + B + D)|X\rangle\langle X| + \frac{1}{2}(A + B - D)|Y\rangle\langle Y| \quad (16)$$

where

$$\begin{aligned} A &= E - \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 + \cos 2\theta) \\ B &= E + \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 - \cos 2\theta) \\ C &= \frac{V_0}{2} \sin 2\theta \\ D &= \sqrt{(A - B)^2 + 4C^2} \end{aligned} \quad (17)$$

and the Eigenstates:

$$\begin{aligned} |X\rangle &= \frac{1}{u}\left(|1\rangle + \frac{1}{2}\frac{e^{i\phi}(B - A + D)}{C}|2\rangle\right), u = \frac{\sqrt{4C^2 + e^{i2\phi}(B - A + D)^2}}{2C} \\ |Y\rangle &= \frac{1}{v}\left(|1\rangle + \frac{1}{2}\frac{e^{i\phi}(B - A - D)}{C}|2\rangle\right), v = \frac{\sqrt{4C^2 + e^{i2\phi}(B - A - D)^2}}{2C} \end{aligned} \quad (18)$$

The flavour states expressed in terms of the Eigenstates in Equation (18) read as:

$$\begin{aligned} |v_e\rangle &= \frac{1}{2D}\left[uJ|X\rangle - vK|Y\rangle\right] \\ |v_\mu\rangle &= \frac{1}{2D}\left[uL|X\rangle + vN|Y\rangle\right] \end{aligned} \quad (19)$$

where, for further convenience, we introduce

$$\begin{aligned} J &= \cos(\theta)(A - B + D) + 2\sin(\theta)C \\ K &= \cos(\theta)(A - B - D) + 2\sin(\theta)C \\ L &= 2\cos(\theta)C - \sin(\theta)(A - B + D) \\ N &= \sin(\theta)(A - B - D) - 2\cos(\theta)C \end{aligned} \quad (20)$$

We then compute the inner products between the chain operators for  $M^1$  and  $M^3$ , and between  $M^2$  and  $M^4$ , respectively, to the equate them to zero in order to achieve the consistency Equation (14). Our aim is to find values of  $V_0$  (and possibly  $\phi$ ) causing the family of histories Equation (11) consistently.

Let us note that the inner products in Equation (13) consist of four components each. To achieve the said consistency Equation (14), one expects at least one of the components of each inner product to equal zero. Note that unless  $t_1 = t_2$ , only one of two components in Equation (13) can vanish. However, the particular value of time instants in the evolution providing consistency of the oscillation's history affects only the probability of the histories, but not the consistency itself. Thus, for the sake of computing, it is not necessary to consider the consistency achieved via time evolution at  $t_1$  or  $t_2$  separately. It is enough to observe that at least one of the four equations below need to be satisfied:

$$\begin{aligned} |u|^2 L(A - B + D) - |v|^2 N(A - B - D)e^{itD} &= 0 \\ |u|^2 J(A - B + D) + |v|^2 K(A - B - D)e^{itD} &= 0 \\ |u|^2 L - |v|^2 Ne^{itD} &= 0 \\ |u|^2 J + |v|^2 Ke^{itD} &= 0 \end{aligned} \quad (21)$$

All the parameters above, except  $u$  and  $v$  (having non-trivial imaginary parts), are real. Thus, the inner products Equation (13) leading to the consistency Equation (14) can be broken down into real and imaginary parts, which are further considered separately. The requirement of the vanishing of the imaginary parts in Equation (21) reads as follows:

$$\begin{aligned} |v|^2 N(A - B - D) \sin(tD) &= 0 \\ |v|^2 K(A - B - D) \sin(tD) &= 0 \\ |v|^2 N \sin(tD) &= 0 \\ |v|^2 K \sin(tD) &= 0 \end{aligned} \quad (22)$$

Let us note that each element of Equation (22) is of a form  $|v|^2 \alpha \gamma \sin(tD)$  where  $\alpha$  is either  $N$  or  $K$  and  $\gamma$  is either  $(A - B - D)$  or 1. Clearly, neither  $v = 0$  as given in Equation (19) nor  $\gamma = 0$ , at least in the presence of normal matter  $V_0 \neq 0$ . To see that  $\alpha$  cannot be zero requires a little more complicated analysis. If  $\alpha = 0$  the corresponding real part for each element is of the form  $|u|^2 \beta \delta$ , where  $\beta$  is either  $L$  or  $J$  and  $\delta$  is either  $(A - B + D)$  or 1. Similar to  $v$  and  $(A - B - D)$ ,  $|u|^2$  and  $(A - B + D)$  cannot equal 0, so  $\delta$  cannot be 0. The only way to achieve consistency, then, is  $\beta = 0$ . As  $L$  with  $N$  and  $J$  with  $K$  appear in pairs, both  $\alpha$  and  $\beta$  being zero would lead to vanishing and, thus, nonphysical flavour states. The imaginary parts in Equation (22) vanish, provided that  $\sin(tD) = 0$  where  $t$  is either  $t_1$  or  $t_2$ , i.e., provided that  $D = \frac{n\pi}{t_1}$  or  $D = \frac{n\pi}{t_2}$  (for  $n \in \mathbb{N}$ ) where  $t_1$ , and  $t_2$  are the time intervals in the set of three-time histories Equation (11). The real parts of Equations (21) should also vanish to allow consistency of histories Equation (11)

$$\begin{aligned}
|u|^2 L(A - B + D) - |v|^2 N(A - B - D) \cos(tD) &= 0 \\
|u|^2 J(A - B + D) + |v|^2 K(A - B - D) \cos(tD) &= 0 \\
|u|^2 L - |v|^2 N \cos(tD) &= 0 \\
|u|^2 J + |v|^2 K \cos(tD) &= 0
\end{aligned} \tag{23}$$

Substituting explicit forms of  $|u|^2$ , and  $|v|^2$  Equations (19)–(23) leads to

$$\begin{aligned}
(B - A + D)L(A - B + D) + (B - A - D)N(A - B - D) \cos(tD) &= 0 \\
(B - A + D)J(A - B + D) - (B - A - D)K(A - B - D) \cos(tD) &= 0 \\
(B - A + D)L + (B - A - D)N \cos(tD) &= 0 \\
(B - A + D)J - (B - A - D)K \cos(tD) &= 0
\end{aligned} \tag{24}$$

and one concludes that the consistency of the set of histories Equation (11) of neutrinos propagating in matter does not depend on  $\phi$ . As vanishing of the imaginary parts of Equation (21) given in Equation (22) holds true provided that  $tD = n\pi$  where  $n \in \mathbb{N}$ , searching for the consistency conditions of Equation (14) can be further split into two cases: (i) for  $n$  even and (ii) odd. For  $n$  even (i) each of Equations (23) becomes solved for  $D = 0$ , which we reject as nonphysical. Therefore, histories in Equation (11) of our system are inconsistent for  $n$  even. On the other hand, for  $n$  odd (ii) the (non-negative) solution to the equations Equation (23) is given by  $V_0 = \frac{\Delta m_{21}^2}{2E}$ . It imposes the consistency conditions,  $V_0 = \frac{\Delta m_{21}^2}{2E}$ ,  $D = 2 \sin \theta \frac{\Delta m_{21}^2}{2E}$ , jointly with  $n = \frac{tD}{\pi}$  being an odd number.

### 3. Discussion

Let us place emphasis on a crucial qualitative difference [21] between histories of vanishing probability and inconsistent families of histories. There are non-trivial histories,  $M$ , for which  $K(M) = 0$ . Such histories are of zero weight, or vanishing probability, and—as they never occur—they are termed as dynamically impossible. On the other hand, inconsistent families of histories are meaningless in the sense that there is no way (at least within consistent history model) to assign probabilities to these histories using time development governed by the laws of quantum dynamics [21]. One might be tempted to say that an inconsistent family of histories never occurs. However, that would be a highly inaccurate description of the physically deep notion of inconsistency [31].

As it is stated earlier in this work, if evolution times  $t_{1,2}$  are not equal  $t_1 \neq t_2$  and  $D$  is either  $\frac{n\pi}{t_1}$ , or  $\frac{n\pi}{t_2}$  the family of histories Equation (11) is consistent for the dynamics generated by Equation (10). Let us note, however, that a particular choice of  $D$  (among  $D$  being  $\frac{n\pi}{t_1}$ , or  $\frac{n\pi}{t_2}$ ) does significantly affect probabilities (weights) of the histories Equation (12). For  $n$  odd,  $D = 2 \sin \theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t_1}$  and  $V_0 = \frac{\Delta m_{21}^2}{2E}$  the family of histories Equation (11) is consistent. This leads to the chain operators  $K(M^i) = 0, i = 1, 2$ . Thus, the probability of both histories  $M^1$ , and  $M^2$  in Equation (11) vanishes. The only non-zero histories in the family are then  $M^{0,3,4}$ . For  $D = \frac{n\pi}{t_2}$  the consistency of Equation (11) is provided with the chain operators  $K(M^i) = 0$  for  $i = 2, 3$ , i.e., probabilities of histories  $M^2$  and  $M^3$  in Equation (11) vanish and the only non-zero histories are then  $M^{0,1,4}$ . For  $t_1 = t_2 = t$  and  $D = \frac{n\pi}{t}$  the family of histories Equation (11) is consistent with  $K(M^i) = 0$  for  $i = 1, 2, 3$  and the only non-zero histories are  $M^{0,4}$ .

The results based upon consistent histories presented here confirm, from a qualitative perspective, the Mikheyev–Smirnov–Wolfenstein conditions. The MSW effect not only shows us that the oscillations for neutrinos propagating in matter differ from those propagating in vacuum, but also provides us with the MSW resonance condition for a maximal probability for change of flavor for neutrino. However, using consistent histories approach as we do, devoid of predicting magnitude of the survival or flavour change probability, we rather aim to search for consistency for a complete family of histories where survival or

change of flavour are both considered with an oscillation to a pure mass state in between. That is why the formalism of consistent histories applied here allows one to investigate the features of the neutrino oscillation problem complementary to the well-established and known results.

#### 4. Conclusions

The consistent histories approach allows one to describe quantum time development as a process that is essentially and always stochastic or probabilistic, and not just when measurements are being carried out. The consistency condition allows one to decide which among the potential histories can occur or can even be considered. We investigated if a class of simple, three-time histories Equation (11) (including quantum property of a neutrino having a definite mass) can be consistent. As a result, we infer whether one can assign a probability to such an ‘event’.

Our analysis showed that, for an overwhelming majority of cases, the set of histories that we considered remained inconsistent and, as a consequence, meaningless. The family of histories we proposed was inconsistent when a neutrino propagated in vacuum. However, under certain tailored dynamical conditions, the family of histories became consistent when a neutrino propagated in the presence of matter, provided that an interaction potential,  $V_0$ , took a very particular value. This value was not only precisely determined by the consistent histories formalism, but also needed to be confronted with known experimental estimates reading for solar neutrinos  $\Delta m_{21}^2 = 1.4906 \times 10^{-6} m_P (8 \times 10^{-5} \text{ eV}^2)$  [38].

The magnitudes of  $V_0$  and  $D$  were extremely small for even a relatively meager value of  $E$  for our analysis. We recalled that the quantity  $D$  represents twice the difference in Eigenvalues for our system and, for such small values of  $D$ , the difference in the Eigenvalues for the system was also extremely small. From this perspective, consistent histories provided additional and supplementary evidence for the neutrino mass states’ being hardly accessible. This result was gained not only from technical viewpoint, as it is in the ‘usual’ quantum mechanics, but also from a fundamental perspective, as it was enforced by the consistent histories approach. Although the Majorana  $CP$ -violation phase does not influence the neutrino oscillation in a 2-flavour case [39], consistency conditions could shed light on neutrinos being Dirac or Majorana type if consistency would depend on it. The consistency of the class of histories investigated in this paper was not affected by the  $CP$ -violating Majorana phase; however, the results presented here, then, hold true for both Majorana or Dirac neutrinos, and would be unchanged after the  $CP$  problem becomes (sooner or later) finally resolved.

In our work, we aimed to present the general properties that lead to the consistency of the proposed family of histories in a general setting for a matter-assisted propagation, not limited to a particular type of neutrino. In our model calculations, we showed that normal matter plays a crucial role in inducing the consistency of certain classes of histories of neutrino oscillations. It suggests that considering vacuum-only models may be, at least for some purposes, an oversimplification. Such an oversimplification may result in excluding—as seemingly meaningless—quantum properties that otherwise can, at least in principle and in a tailored parameter regime, be consistently studied.

Quantum formalism based on consistent histories is of great predictive power, as it can be applied to gain a useful knowledge concerning quantum systems without measuring them. There are quantitative properties that can be calculated using consistent formalism and are inaccessible in a standard quantum approach. Probabilities (weights) of multi-time histories going beyond a standard Born rule are probably the best examples thereof. On the other hand, consistent histories, such as a quantum formalism, allow one to investigate the qualitative features of quantum evolution, such as the consistency of certain histories. As considering the consistency, per se, goes beyond standard quantum formalism, all the results obtained in our work are essentially novel with respect to that known about neutrino oscillation so far. The qualitative results concerning the consistency of certain histories presented in our work were supplemented with quantitative predictions on consistency-

granting parameters. Let us emphasise that, at least in our work, a primary advantage is using consistent histories as a predictive tools for the qualitative properties of neutrino oscillations. We hope that our contribution may motivate further studies of neutrino properties with consistent histories' formalism as a useful complementary tool.

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## 9 (Article D) Consistency of Histories of Neutrino Oscillation in the Presence of Normal Matter and Continuous Non-Selective Measurement

### 9.1 Introduction

The aim of Article D was to study the impact of a measurement like mechanism on the consistency of a family of histories. For continuity, the system was neutrino oscillations in the presence of normal matter. The family of history too was as that in Article C. It is a three-time history. The first and third events correspond to the neutrino being in a flavor state, while the second event places the neutrino in a mass state [D29].

Article C showed that presence of matter is essential in making this family of histories in the article consistent. Article D however considers the role of Decoherence induced by a mechanism that mimics a continuous non-selective measurement of an “observable” analogous to a hypothetical measurement of neutrino mass [D20]. The impact of continuous non-selective measurement on the neutrino oscillation itself was documented in Article A. Thus, Article D combines the studies in Articles A and C to explore the impact of such a measurement mechanism on the consistency of family of histories.

Article D treats the system as an open system, and considers a specific type of Decoherence arising from a non-selective measurement like quantum-classical environment affecting the oscillations [D17], thus investigating the impact of back-effect from continuous non-selective measurement on the consistency of the family of histories.

### 9.2 Model

Article D employs the Qubit Approximation of Neutrinos covered in Chapter 2. As in Article C, there are two orthogonal mass states  $|1\rangle$  and  $|2\rangle$ , and two flavor states  $|v_e\rangle$  and  $|v_\mu\rangle$ . The dynamics of this simplified two-flavor Majorana type (hypothetical) neutrino propagating in ordinary matter is represented by the Neutrino Wolfenstein effective Hamiltonian

[D1,D4], presented in Chapter 2.4.

$$H = H_0 + \frac{V_0}{2} \begin{pmatrix} 1 + \cos 2\theta_{12} & e^{-i\phi} \sin 2\theta_{12} \\ e^{i\phi} \sin 2\theta_{12} & 1 - \cos 2\theta_{12} \end{pmatrix},$$

where  $V_0$  is the effective interaction potential due to oscillation of the neutrino in the presence of normal matter.

This Hamiltonian represents the Hamiltonian for the "system"  $H_{sys}$  for the total system-probe Hamiltonian discussed in Chapter 3.2,

$$H_{sys+probe} = H_{sys} + H_{probe} + g(t)H_{int}$$

where observable  $A$  of a quantum system couples with probe position  $Q$  to give the interaction term  $H_{int} = A \otimes Q$ .  $g(t)$  is the strength of the coupling. The environment is modelled to mimic a Continuous non-selective measurement like action, resulting in a quantum master equation from Chapter 3.2,

$$\frac{d}{dt}\rho(t) = -i[H_{sys}, \rho(t)] - \frac{\sigma}{2}[A, [A, \rho(t)]].$$

The first term represents the information extracted due to the measurement process, and the second represents the back-action  $\sigma$  on the state of the system due to the measurement [D31,D32].

$A$  is the observable coupled with the environment is the hypothetical measurement of mass state of a neutrino,  $M$ , equation 8 of the article

$$M = \sum_{x=1,2} a_x \Pi_x, \text{ where } \Pi_x = |v_x\rangle\langle v_x|, x = 1, 2$$

where  $a_1 = -a_2$  [D17].

The family of histories in the article is equation 15 of the article.

$$M^0 : [v_\mu] \odot [I] \odot [I]$$

$$M^1 : [v_e] \odot [1] \odot [v_\mu]$$

$$M^2 : [v_e] \odot [1] \odot [v_e]$$

$$M^3 : [v_e] \odot [2] \odot [v_\mu]$$

$$M^4 : [v_e] \odot [2] \odot [v_e],$$

where  $[x] = |x\rangle\langle x|$ , is the projector for finding neutrino is state  $|x\rangle$ .

As discovered in Article C, the chain operators for this family of histories are mutually exclusive apart from those for  $M^1$  with  $M^3$ , and for  $M^2$  with  $M^4$ , and solving for consistency is thus finding dynamical parameters that lead to

$$\begin{aligned}\langle K(M^1) | K(M^3) \rangle &= 0 \\ \langle K(M^2) | K(M^4) \rangle &= 0\end{aligned}$$

In the absence of back-action of measurement,  $\sigma = 0$ , article C showed that this family of histories is inconsistent in vacuum, but in the presence of normal matter, when  $V_0 = \frac{\Delta m_{21}^2}{2E}$  and the parameter  $D = \frac{n\pi}{t}$ , the family becomes consistent [D29].

Article D however has a total system-probe Hamiltonian instead. Using the parametrization below:

$$\begin{aligned}A &= E - \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 + \cos 2\theta) \\ B &= E + \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 - \cos 2\theta) \\ C &= \frac{V_0}{2} \sin 2\theta,\end{aligned}$$

the qubit neutrino can be expressed as:

$$H_{sys} = \begin{pmatrix} A & e^{-i\phi}C \\ e^{i\phi}C & B \end{pmatrix}$$

$D$ , which is  $\lambda_1 - \lambda_2$ ,  $\lambda_{1,2}$  being eigenvalues of  $H_{sys}$ , can then be expressed as

$$D = \sqrt{(A - B)^2 + 4C^2}.$$

The mass observable being measured can be expressed as follows in the mass state basis:

$$M = |1\rangle\langle 1| - |2\rangle\langle 2| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

### 9.3 Results

The master equation is rewritten in the form  $\dot{\rho} = \hat{\mathcal{L}}\rho$ , where  $\hat{\mathcal{L}}$  is the Liouvillian superoperator. This is further parameterized to Bloch space, resulting

in the following expression for  $\mathcal{L}$ :

$$\mathcal{L} = \begin{pmatrix} -2\sigma^2 & B - A & 2C\sin\phi \\ A - B & -2\sigma^2 & -2C\cos\phi \\ -2C\sin\phi & 2C\cos\phi & 0 \end{pmatrix}$$

rendering the solution of the Master Equation to be that of solving equation 23 of the article,  $\langle \dot{\sigma} \rangle = \mathcal{L}\langle \sigma \rangle$ .

Article D did not compute the states of the system under this dynamics, or transition probability between states, and therefore did not require the full solution of the Master Equation.  $e^{\mathcal{L}t}$  was computed to obtain the time development operator  $T = e^{\mathcal{L}t}$ . The dynamics in Article C could be solved analytically, however, the inclusion of the back-effect from continuous, non-selective measurement deemed it otherwise. *SageMath* was used for all computations for the article.

Eigenvalues for the characteristic equation for  $\mathcal{L}$  were computed, and using *Cayleigh–Hamilton* Theorem, that is that each matrix satisfies its own characteristic equation, and that *exponential* is an analytic function,  $e^{\mathcal{L}t}$  was calculated as being:

$$e^{\mathcal{L}t} = \alpha_0 \mathbb{I} + \alpha_1 \mathcal{L} + \alpha_2 \mathcal{L}^2$$

where  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$  satisfy the system of equations

$$e^{\lambda_i t} = \sum_{k=0}^2 \alpha_k \lambda_i^k$$

There are two intervals between the three events in each history:  $t_1$  corresponds to the time interval between the initial pure flavor state and the pure mass state, and  $t_2$  represents the time interval between the pure mass state and the pure flavor state. The choice of  $t_1$  or  $t_2$  to achieve consistency influences the probabilities of the histories but does not affect the solution itself. Therefore, for the computations, the article considered only one time interval,  $t$ . Thus, consistency for this family of histories is achieved if

$$T[v_e] = [1] \text{ or } T^\dagger[1] = [v_\mu].$$

For each of these two consistency equations, there are different set of values for the parameters  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$ . However, these parameters must also

satisfy the system of equations derived from  $e^{\lambda_i t} = \sum_{k=0}^2 \alpha_k \lambda_i^k$ . Equations 35 and 36 of the article show the set of values for  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$  for  $T[v_e] = [1]$  and  $T^\dagger[1] = [v_\mu]$  respectively, and consistency is obtained if the multivariate function  $f(V_0, \sigma^2) = 0$ , Equation 34 of the article.

The article further parameterizes and scales the time  $t$  in units of  $E$  as  $V_0 = \dot{V} \frac{\Delta m_{21}^2}{2E}$ ,  $\sigma^2 = \dot{\sigma}^2 \frac{\Delta m_{21}^2}{2E}$ , and  $t = \dot{t} \frac{2E}{\Delta m_{21}^2}$ . This renders  $\dot{V}$  and  $\dot{\sigma}^2$  dimensionless and  $\frac{\sqrt{3}}{6}(X+y)\dot{t}$  is in radians, and factors out  $\frac{\Delta m_{21}^2}{2E}$  from the equation. Thus, the consistency equation becomes  $f(\dot{V}, \dot{\sigma}^2) = 0$ .

Figure 1 of the article shows the minimum absolute value of  $f$  for values of  $\dot{V}$  for  $\dot{\sigma}^2 \geq 0$ , where  $T[v_e] = [1]$ . It shows that  $f(\dot{V}, \dot{\sigma}^2) = 0$  for negative values of  $\dot{V}$ ,  $\approx -1.3 \leq \dot{V} \leq -1$ , that is  $\approx -1.3 \frac{\Delta m_{21}^2}{2E} \leq V_0 \leq -1 \frac{\Delta m_{21}^2}{2E}$ . This is rejected as nonphysical.

Figure 2 of the article shows the minimum absolute value of  $f$  for values of  $\dot{V}$  for  $\dot{\sigma}^2 \geq 0$ , where  $T^\dagger[1] = [v_\mu]$ . It shows that  $f(\dot{V}, \dot{\sigma}^2) = 0$  in the range  $1 \leq \dot{V} \leq \approx 1.24$ , that is  $\frac{\Delta m_{21}^2}{2E} \leq V_0 \leq \approx 1.24 \frac{\Delta m_{21}^2}{2E}$ . Figure 3 plots the values of  $\dot{V}$  and  $\dot{\sigma}^2$  where  $f(V_0, \sigma^2) = 0$  for  $T^\dagger[1] = [v_\mu]$  and Table 1 further lists the corresponding values for  $\frac{\sqrt{3}}{6}(X+y)\dot{t}$ .

The family of histories studied in the article is consistent if  $T[v_e] = [1]$  or  $T^\dagger[1] = [v_\mu]$ , but both cannot be satisfied simultaneously, as they are mutually exclusive events in Consistent Histories Formalism. For the former, similar to article 3 where there is no back-effect from a measurement like mechanism, the family of histories is consistent for negative values of  $\dot{V}$ , and consequently  $V_0$ , which is rejected. For  $T^\dagger[1] = [v_\mu]$ , however, this family of histories is consistent for the range of values in Table 1.

The introduction of the back-effect from the environment, mimicking continuous non-selective measurement of the ‘Mass’ Observable, leads to a broader set of values that achieve consistency, compared to the case without this *measurement*.

## 9.4 Conclusion

Article D studied the same family of histories as Article C, composed of quantum events where a qubit approximation of neutrino starting in an

electron flavor ends up in one of the mass states after some evolution in time, and then further evolves in time and gets back to one of the flavor states. This family of histories is consistent. Article C showed that it is consistent for a particular value of the interaction potential and the time interval(s) between *events*.

Article A studied neutrino oscillations in the presence of an environment that mimics continuous non-selective measurement which leads to the suppression of the neutrino oscillations to an imperfect Zeno-like mechanism. The family of histories remains inconsistent for the majority of the range of values of the dynamics, but the back-effect from continuous non-selective measurement leads to the expansion of parameters of the dynamics of the evolution, and consistency is achieved for a range of values instead of one particular value that is the case for without this measurement.

## Article

# Consistency of Histories of Neutrino Oscillation in the Presence of Normal Matter and Continuous Non-Selective Measurement

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**Abstract:** Two-flavor neutrino oscillation is analyzed using the consistent histories approach. We identify the properties of neutrinos and the oscillation conditions, such as the presence of matter or the impact of measurement, that ensure the consistency of three-time neutrino histories. The connection between the consistency of these histories and the well-known CP violation explored in this study serves as a specific example.

**Keywords:** neutrino oscillation; consistent histories; CP violation; quantum open systems; continuous measurement

## 1. Introduction

Neutrinos, often hypothesized as key to understanding the Universe and its origins, are studied across various disciplines. At the heart of this research are the fascinating dynamical properties of neutrino oscillations [1–3], which, while naturally investigated within the realm of particle physics [4], have also garnered interest from the perspectives of decoherence [1,5–10] and quantum information processing [2,11–17]. Sophisticated experiments [18] have illuminated some of these properties, yet many fundamental questions about the time evolution of neutrinos [4,19] and their oscillations remain unresolved.

As neutrinos hardly interact with anything, they have legitimately become a template for examples and studies concerning systems resistant to measurements, and open up further avenues of research. In the absence of a measurement scheme of the mass of a neutrino, one can adopt the typical *terminology* associated with quantum measurement [20,21], and study neutrino oscillations in a model that mimics *Continuous Non-Selective* measurement by environment [17].

Alternatively, one can move beyond the orthodox interpretations of measurement and employ the *Consistent* (or *Decoherent*) Histories Formalism to extract information about systems where direct measurements are unavailable due to fundamental or technical limitations [22–26]. A quantum history of a physical system consists of a sequence of quantum events at successive times, where a quantum event at a given time represents any quantum property of the system in question [22]. For instance, a neutrino being in a specific mass state constitutes such an event. If a *family* of these histories is shown to be *consistent*, probabilities can be assigned to the occurrence of these events. Conversely, one can investigate the properties or parameters necessary to render a family of histories consistent.

In this paper, following a brief review of neutrino oscillations, continuous non-selective measurement, and the consistent history approach, we analyze the impact on the consistency of a family of histories for a neutrino oscillating in the presence of matter. Specifically,



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we consider the role of decoherence induced by a mechanism that mimics a continuous non-selective measurement [20] of an “observable” analogous to a hypothetical measurement of neutrino mass.

We consider a simplified two-flavor model of neutrino oscillations, focusing on electron–muon oscillations [1,4,19,27,28]. Despite its simplicity, this model provides valuable insights into the dynamics of neutrino oscillations. Additionally, we examine three time histories of the two-flavor neutrino evolution. These histories represent the shortest non-trivial sequences that cannot be analyzed using the standard Born rule. Each three-time history is so named because it consists of three “events”. In our analysis, the first and third events correspond to the neutrino being in a flavor state, while the second event places the neutrino in a mass state [29].

The presence of matter is crucial for achieving the consistency of such three-time families of histories in neutrino oscillations [29], particularly when there is a scattering-like interaction between neutrinos and ordinary matter. In this paper, we treat the system as an open quantum system, where a specific type of decoherence arises from a hypothetical *non-selective* measurement-like quantum-classical environment, which affects the oscillations [17]. While potential CP violation from a hypothetical non-vanishing Majorana phase does not influence the consistency of the histories considered [29], our focus is on investigating how the back-effect from continuous non-selective measurements impacts the consistency of these histories. Additionally, we explore whether this back-effect introduces a dependence on the consistency of the Majorana phase.

## 2. Qubit Approximation of Neutrinos

The qubit approximation utilizes a two-level system, where a neutrino can exist in two flavor states  $v_e, v_\mu$ , which do not directly correspond to the two mass states  $v_1, v_2$ , allowing for simpler mixing parameters and often enabling analytical solutions. The flavor states are expressed as  $|v_e\rangle = \cos\theta |1\rangle + e^{i\phi} \sin\theta |2\rangle$  and  $|v_\mu\rangle = -\sin\theta |1\rangle + e^{i\phi} \cos\theta |2\rangle$ , where  $\theta = \theta_{12}$  is the mixing angle. The phase  $\phi$  represents the CP-violating Majorana phase, which vanishes under the  $U(1)$  global transformation in the Dirac phase but remains non-zero in the Majorana case [4].

The vacuum Hamiltonian for a two-level relativistic neutrino [4] is given by

$$H_0 = \begin{pmatrix} E - \frac{\Delta m_{21}^2}{4E} & 0 \\ 0 & E + \frac{\Delta m_{21}^2}{4E} \end{pmatrix} \quad (1)$$

where  $\Delta m_{21}^2 = m_2^2 - m_1^2$  is the mass-squared difference in the normal hierarchy case, and  $E$  is the energy of the massless neutrino. The effective Hamiltonian for a neutrino propagating in normal matter, known as the Wolfenstein effective Hamiltonian [1,4], is given by the following:

$$H_F = H_0 + \frac{V_0}{2} \begin{pmatrix} 1 + \cos 2\theta & e^{-i\phi} \sin 2\theta \\ e^{i\phi} \sin 2\theta & 1 - \cos 2\theta \end{pmatrix} \quad (2)$$

where  $N_e$  is the electron density,  $G_F$  is the Fermi constant, and  $V_0 = \sqrt{2}G_F N_e$  is the effective interaction potential due to the oscillation of the neutrino in the presence of normal matter, arising from coherent forward scattering on electrons via charged current interactions. The neutral current interaction does not contribute effectively to the Hamiltonian as per the Standard Model [4].

### 3. Continuous Non-Selective Measurement

Quantum measurements are fundamentally different from the classical concept of measurement. Real experiments attempting to measure physical quantities can rarely be fully described by the von Neumann scheme [20,21,30], in which the system being measured and the apparatus performing the measurement *interact* to produce what we refer to as a *measurement*.

For quantum systems, measurement is not direct and can be thought of as a two-step process. The system being measured interacts with another system that acts as a measurement *probe*. This probe, as a system, has properties that can be read by a classical apparatus. Thus, an observable  $A$  of the quantum system couples to the probe's *position*  $Q$ , resulting in an interaction between the system and the probe. Next, the probe's *momentum*  $P$ , canonically conjugated to  $Q$ , is projectively measured. The first part of this process can be viewed as the *act of making the measurement*, while the second part represents the *reading of the measurement made* [20,21,30].

A continuous measurement involves the continual extraction of information from a system, whereas a non-selective measurement is one where the result is ignored or *not read* [20]. Therefore, we treat the interaction with the environment as a neutrino propagates in the presence of normal matter as a continuous measurement-like interaction, where the result is not read [17]. The first part of the measurement is described by the system-probe total Hamiltonian:

$$H_{sys+probe} = H_{sys} + H_{probe} + g(t)H_{int} \quad (3)$$

where  $g(t)$  is the strength of the coupling. The observable  $A$  of a quantum system couples with the probe's position  $Q$  to give the interaction term  $H_{int} = A \otimes Q$ .

Assuming a non-selective measurement over a time interval divided into intervals of length  $q$ , the dynamics of the system, after being reduced with respect to the probe, are no longer unitary; however, they form a completely positive quantum map. For a set of measurement operators  $E_x$ , where  $\sum_x E_x^\dagger E_x = \mathbb{I}$ , the non-selective measurement of a system in state  $\rho$  is the transformation  $\rho \rightarrow E_x \rho E_x^\dagger$ , which is an example of a completely positive map [31].

The evolution of the state due to such a measurement on each time interval of length  $q$  is given by the following:

$$\rho(t+q) = \int dp E_p \rho(t) E_p^\dagger \quad (4)$$

with  $E_p = e^{iqH_{sys}} \langle p | e^{-iqH_{int}} | \phi_q \rangle$ . Here,  $|p\rangle \in H_{probe}$  is an eigenstate of  $P$ , the probe momentum, and  $|\phi_q\rangle \in H_{probe}$  is the state of the probe before the measurement. Expanding Equation (4) to the second order in  $q$  for a finite measurement duration leads to a quantum master equation:

$$\frac{d}{dt}\rho(t) = -i[H_{sys}, \rho(t)] - \frac{\sigma}{2}[A, [A, \rho(t)]] \quad (5)$$

where the first term represents the extraction of information due to the measurement process, and the second term represents the back-action on the state of the system due to the measurement [31,32].

The parameter  $\sigma$  describes the back-action of the measurement. For an integrated coupling strength proportional to the length of the time interval  $q \sim \int_0^t g(s)ds$ ,  $\sigma$  is given by the following:

$$\sigma^2 = \lim_{q \rightarrow 0} q \langle \phi_q | Q^2 | \phi_q \rangle \quad (6)$$

where the limit describes the transition from a step-like to a truly continuous measurement.

We consider a continuous non-selective measurement of a neutrino to be described by the master Equation (5), where

$$H_{sys} = H_F \quad (7)$$

and the “observable” that is hypothetically measured corresponds to a *mass state measurement*, diagonal in the mass state basis,

$$M = \sum_{x=1,2} a_x \Pi_x; \text{ where; } \Pi_x = |x\rangle\langle x|; x = 1, 2 \quad (8)$$

where  $a_1 = -a_2$  [17].

Selective measurements of neutrinos are rare and challenging, but this model presents a *close to non-selective* case, which increases the probability of transition from the electron flavor to the muon flavor, and even more so for CP-violating systems [17].

#### 4. Consistent Histories Formalism for Qubit Approximation of Neutrino Oscillation

The Consistent Histories approach treats measurement beyond the traditional Copenhagen interpretation of Quantum Mechanics. It defines a *quantum event* as any quantum property of the system at a particular time, and a *quantum history* as a sequence of such quantum events at successive times [22], making each history a compound quantum event that represents a specific time evolution of the system. The time evolution of the system is treated as a stochastic process, meaning that the past or future states of the system are not determined by the present state.

Each quantum event is mathematically represented as a projection operator  $P$  corresponding to the quantum property that defines the event. Thus, a quantum history, analogous to a stochastic process, of length  $n$  is a sequence of projectors  $P_i$ , with one projector for each time  $t_i$ , where  $i = 0, \dots, (n-1)$ . Since the process is akin to a stochastic process, the events come from a well-defined event algebra, which is obtained by considering quantum events that are mutually exclusive to each other, with the projectors representing these events summing up to the identity.

$$P^j P^k = \delta_{jk} P^k, \quad \sum_j P^j = I \quad (9)$$

In other words, one does not consider just a single history, but a *complete family of histories* [22,33]. Thus, these families of histories are composed of projective decompositions of the identity,  $PDI = P^k$ , which is referred to as a *framework*, and must originate from a single framework in order to include only compatible properties, as per the *single framework rule* [22,33–35].

At each time instant  $t_i$ , all the quantum properties of the system correspond to elements of a  $PDI(t_i)$ , with each projector mapping quantum states onto subspaces of the *instantaneous* Hilbert Space  $\mathcal{H}(t_i)$ . The space for a history is mathematically the tensor product of the instantaneous Hilbert Spaces:

$$\check{\mathcal{H}} = \bigodot_{i=0}^n \mathcal{H}(t_i) \quad (10)$$

where  $\odot$  represents the temporal composite (multi-time), in contrast to  $\otimes$ , which is used for spatial composites (multi-partite) [22].

Each history  $M^k$  can then be expressed as

$$M^k = \bigodot_{i=0}^n P^{(i,k)} \quad (11)$$

where  $P^{(i,k)} \in PDI(t_i)$ . The consistency condition involves assigning probabilities, or weights, to non-commuting quantum properties only if there is no interference between pairs of histories that should be decoherent. Weights for histories are derived by extending the Born rule to multi-time histories [22,23,33–35], and are computed as

$$W(M^k) = \langle K(M^k), K(M^k) \rangle = \text{Tr} [K^\dagger(M^k) K(M^k)] \quad (12)$$

where for a time evolution operator  $T(t_i, t_j)$ ,  $K(M^k)$  is called the history's chain operator [22].

$$K(M^k) = P^{(n,k)} \prod_{i=0}^{n-1} T(t_{i+1}, t_i) P^{(i,k)} = P_n T(t_n, t_{n_1}) P_{n-1} \cdots P_1 T(t_1, t_0) P_0 \quad (13)$$

The decoherence of histories comes from the *consistency condition* [22,23,33–35], which is formulated as follows:

$$\langle K(M^k), K(M^s) \rangle = 0, \text{ for all } k \neq s \quad (14)$$

There is a qualitative distinction between histories with zero weight (vanishing probability) and inconsistent families of histories [22]. Histories  $M$  such that  $K(M) = 0$  represent events that do not occur and are referred to as *dynamically impossible histories*. Inconsistent families of histories have no meaning within the consistent histories framework, and no weight or probability can be assigned to such families through time evolution governed by the laws of quantum dynamics [22,35]. It is also possible to write families of histories that are inherently inconsistent, regardless of the time evolution governing the system. However, of greater interest are families of histories that are largely inconsistent unless the dynamics that evolve the system over time adopt certain parameters, where solving the consistency condition is equivalent to identifying those parameters.

One such example is the one explored in [29], a family of three-time histories which is also the one we study further in this paper.

$$\begin{aligned} M^0 &: [v_\mu] \odot [I] \odot [I] \\ M^1 &: [v_e] \odot [1] \odot [v_\mu] \\ M^2 &: [v_e] \odot [1] \odot [v_e] \\ M^3 &: [v_e] \odot [2] \odot [v_\mu] \\ M^4 &: [v_e] \odot [2] \odot [v_e] \end{aligned} \quad (15)$$

where  $[x] = |x\rangle\langle x|$ ,  $x = \nu_{e,\mu}, 1, 2$ . All histories begin with the neutrino in the electron flavor state, except for  $M^0$ , which is required to obtain the PDI. At the second time instance, the histories place the neutrino in a mass state, followed by a return to a flavor state at the third time instance. The history  $M^0$  can be expanded into four similar histories, where the initial state is a muon flavor neutrino, evolving to a mass state at the second time instance, and returning to a flavor state at the third, much like the histories for the electron neutrino, with the same results [29].

Utilizing Equation (13), the chain operators for the family of histories are obtained as follows:

$$\begin{aligned} K(M^0) &= T_2 T_1 |v_\mu\rangle\langle v_\mu| \\ K(M^1) &= \langle v_\mu | T_2 | 1 \rangle \langle 1 | T_1 | v_e \rangle \cdot |v_\mu\rangle\langle v_e| \\ K(M^2) &= \langle v_e | T_2 | 1 \rangle \langle 1 | T_1 | v_e \rangle \cdot |v_e\rangle\langle v_e| \\ K(M^3) &= \langle v_\mu | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \cdot |v_\mu\rangle\langle v_e| \\ K(M^4) &= \langle v_e | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \cdot |v_e\rangle\langle v_e| \end{aligned} \quad (16)$$

where  $T_1 = \exp[i(t_0 - t_1)H]$  and  $T_2 = \exp[i(t_1 - t_2)H]$  represent the time evolution operators for the first and second intervals of the three-time histories. By applying the consistent histories Equation (14) to each pair of histories in the family, all histories are mutually exclusive to each other, except for  $M^1$  versus  $M^3$ , and  $M^2$  versus  $M^4$  [29], which are given as follows:

$$\begin{aligned} \langle K(M^1) | K(M^3) \rangle &= \langle 1 | T_2^\dagger | v_\mu \rangle \langle v_e | T_1^\dagger | 1 \rangle \langle v_\mu | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \\ \langle K(M^2) | K(M^4) \rangle &= \langle 1 | T_2^\dagger | v_e \rangle \langle v_e | T_1^\dagger | 1 \rangle \langle v_e | T_2 | 2 \rangle \langle 2 | T_1 | v_e \rangle \end{aligned} \quad (17)$$

The case in which there is no back-effect from the environment and the time evolution operators  $T$  are derived from  $H = H_F$  as given in Equation (2) was solved in [29] by determining the parameters that make

$$\begin{aligned} \langle K(M^1) | K(M^3) \rangle &\equiv 0 \\ \langle K(M^2) | K(M^4) \rangle &\equiv 0 \end{aligned} \quad (18)$$

and render the family of histories consistent. For these dynamics, the family of histories is inconsistent in a vacuum, but in the presence of normal matter, where  $V_0 = \frac{\Delta m_{21}^2}{2E}$  and the parameter  $D = \frac{n\pi}{E}$ , the family becomes consistent. Here,  $D$  is defined as the difference between the eigenvalues of  $H_F$ , specifically  $D = \lambda_1 - \lambda_2$  [29].

In this paper, we consider a two-flavor neutrino oscillating in the presence of normal matter, where the environment simulates a continuous non-selective measurement of the observable  $M$  from Equation (8). This interaction introduces the back-action  $\sigma$  in the quantum master Equation (5), with  $H_{sys} = H_F$ , allowing us to derive the time evolution operator  $T$  for this process. We then solve Equation (18) to identify the dynamical parameters that make the family of histories (15) consistent under these dynamics. We demonstrate that the additional effect of the environment simulating a continuous non-selective measurement of the “mass observable” expands the range of parameter values that lead to consistency.

## 5. Method

In the mass basis, the qubit neutrino can be expressed as

$$H_{sys} = \begin{pmatrix} A & e^{-i\phi}C \\ e^{i\phi}C & B \end{pmatrix} \quad (19)$$

where

$$\begin{aligned} A &= E - \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 + \cos 2\theta) \\ B &= E + \frac{\Delta m_{21}^2}{4E} + \frac{V_0}{2}(1 - \cos 2\theta) \\ C &= \frac{V_0}{2} \sin 2\theta \end{aligned} \quad (20)$$

$D$ , which is  $\lambda_1 - \lambda_2$ ,  $\lambda_{1,2}$  being eigenvalues of  $H_{sys}$ , can then be expressed as

$$D = \sqrt{(A - B)^2 + 4C^2}.$$

The mass observable being measured can be expressed as follows:

$$M = |1\rangle\langle 1| - |2\rangle\langle 2| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (21)$$

To solve the master equation, we rewrite it in the form  $\dot{\rho} = \hat{\mathcal{L}}\rho$ , where  $\hat{\mathcal{L}}$  is the Liouvillian superoperator. We then parameterize the system in Bloch space, resulting in the following expression for  $\mathcal{L}$ :

$$\mathcal{L} = \begin{pmatrix} -2\sigma^2 & B - A & 2C\sin\phi \\ A - B & -2\sigma^2 & -2C\cos\phi \\ -2C\sin\phi & 2C\cos\phi & 0 \end{pmatrix} \quad (22)$$

The solution to Equation (5) then comes from solving the equation below:

$$\langle \dot{\sigma} \rangle = \mathcal{L} \langle \sigma \rangle \quad (23)$$

However, as we do not need to compute the states of the system, but just the time evolution dynamics to use for checking consistency, we do not need a full solution to the master equation, but just compute  $e^{\mathcal{L}t}$  to obtain the time development operator  $T = e^{\mathcal{L}t}$ . Having  $T$ , we substitute it to Equation (17) and solve for Equation (18).

Computing  $e^{\mathcal{L}t}$  requires determining the eigenvalues of  $\mathcal{L}$ . While the matrix  $H_F$  could be solved analytically, the introduction of the back-effect from the continuous non-selective measurement necessitates solving for  $\mathcal{L}$  computationally. All computations for this paper were performed using *SageMath* (<https://www.sagemath.org/>). The characteristic equation for the eigenvalues of  $\mathcal{L}$  is given by the following:

$$\lambda^3 + 4\sigma^2\lambda^2 + (D^2 + 4\sigma^4)\lambda + 8C^2\sigma^2 = 0 \quad (24)$$

The eigenvalues are thus the solutions to Equation (24), computed through the SageMath eigenvalue function, which turns out to be

$$\begin{aligned} \lambda_1 &= \frac{1}{3}[y - X - 4\sigma^2] \\ \lambda_2 &= \frac{1}{6}[(X - y - 8\sigma^2) - i\sqrt{3}(X + y)] \\ \lambda_3 &= \frac{1}{6}[(X - y - 8\sigma^2) + i\sqrt{3}(X + y)] \end{aligned} \quad (25)$$

where we define

$$\begin{aligned} y^3 &= 8\sigma^6 + 18\sigma^2[(A - B)^2 - 2C^2] + \sqrt{432\sigma^8(A - B)^2 + 27D^6 + 216\sigma^4[(A - B)^4 - 10(A - B)^2C^2 - 2C^4]} \\ X &= \frac{3D^2 - 4\sigma^4}{y} \end{aligned} \quad (26)$$

As per the *Cayleigh–Hamilton* Theorem, each matrix satisfies its own characteristic equation, therefore,

$$\Delta(\mathcal{L}) = \mathcal{L}^3 + \sigma^2 \mathcal{L}^2 + (D^2 + 4\sigma^4)\mathcal{L} + 8C^2\sigma^2 = [0] \quad (27)$$

Any analytic function  $f(A)$  of a matrix  $A$  can be written as  $f(A) = \sum_{k=0}^{\infty} \alpha_k A^k$ , which can be written as a polynomial as  $f(s) = \sum_{k=0}^{\infty} \alpha_k s^k$ . Per *Remainder Factor* Theorem,  $f(s) = \Delta(s)Q(s) + R(s)$ , where  $\Delta(s)$ ,  $Q(s)$ , and  $R(s)$  are the divisor, quotient, and remainder, respectively. However, for  $s = \lambda$ , we obtain

$$f(\lambda) = \Delta(\lambda)Q(\lambda) + R(\lambda) = R(\lambda) = \sum_{k=0}^{n-1} \alpha_k \lambda^k \quad (28)$$

Similarly,  $f(A) = \Delta(A)Q(A) + R(A)$ , for a matrix  $A$ . By the *Cayleigh–Hamilton* Theorem,  $\Delta(A) = 0$ . Therefore,

$$f(A) = \sum_{k=0}^{n-1} \alpha_k A^k \quad (29)$$

such that  $\alpha_k$  satisfy  $f(\lambda_i) = \sum_{k=0}^{n-1} \alpha_k \lambda_i^k$ . Therefore,  $\mathcal{L}$  being a  $3 \times 3$  matrix, and *exponential* being an analytic function, by Equation (28),

$$e^{\mathcal{L}t} = \alpha_0 \mathbb{I} + \alpha_1 \mathcal{L} + \alpha_2 \mathcal{L}^2 \quad (30)$$

where  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$  satisfy the system of equations

$$e^{\lambda_i t} = \sum_{k=0}^2 \alpha_k \lambda_i^k \quad (31)$$

Additionally,  $e^{\mathcal{L}t}$  will be used to solve Equation (18), with the full forms of the consistency condition provided in Equation (17). There are two intervals between the three events in each history:  $t_1$  corresponds to the time interval between the initial pure flavor state and the pure mass state, and  $t_2$  represents the time interval between the pure mass state and the pure flavor state. To obtain a solution, we consider only one time  $t$ . The choice of  $t_1$  or  $t_2$  to achieve consistency influences the probabilities of the histories but does not affect the solution itself. Thus, the solution to Equation (18), which ensures the consistency of this family of histories (15), can be expressed as follows:

$$T[v_e] = [1] \text{ or } T^+[1] = [v_\mu] \quad (32)$$

Each of the two consistency equations in (32) provides a different set of values for the parameters  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$ . However, these parameters must also satisfy the system of equations derived from  $e^{\lambda_i t} = \sum_{k=0}^2 \alpha_k \lambda_i^k$ . We define

$$z = \frac{\sqrt{3}(X+y)[3\alpha_1 + (X-y-8\sigma^2)\alpha_2]}{18\alpha_0 + 3(X-y-8\sigma^2)\alpha_1 - (X^2 + 4Xy + y^2 + 8\sigma^2(X-y) - 32\sigma^4)\alpha_2} \quad (33)$$

Thus, solving the set of equations parameters  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$  can be simplified into one multi-variate equation below:

$$f(V_0, \sigma^2) = \frac{t}{3} [y - X - 4\sigma^2] - \ln \left[ \frac{1}{9} (9\alpha_0 + 3(y - X - 4\sigma^2)\alpha_1 + (y - X - 4\sigma^2)^2\alpha_2) \right] = 0 \quad (34)$$

such that  $\frac{\sqrt{3}}{6}(X + y)t = \tan^{-1}(z) \neq 0$ .

For  $T[v_e] = [1]$ ,

$$\begin{aligned} \alpha_0 &= \frac{8\sigma^4(C\cos 2\theta - (A - B)\sin 2\theta)\sin 2\theta}{8\sigma^4(C\cos 2\theta - (A - B)\sin 2\theta)(\sin 2\theta \cos 2\theta) - ((A - B)\cos 2\theta + 2C\sin 2\theta)(2C\cos 2\theta - (A - B)\sin 2\theta)^2} \\ \alpha_1 &= \frac{4\sigma^2(C\cos 2\theta - (A - B)\sin 2\theta)\sin 2\theta}{8\sigma^4(C\cos 2\theta - (A - B)\sin 2\theta)(\sin 2\theta \cos 2\theta) - ((A - B)\cos 2\theta + 2C\sin 2\theta)(2C\cos 2\theta - (A - B)\sin 2\theta)^2} \\ \alpha_2 &= \frac{(2C\cos 2\theta - (A - B)\sin 2\theta)\sin 2\theta}{8\sigma^4(C\cos 2\theta - (A - B)\sin 2\theta)(\sin 2\theta \cos 2\theta) - ((A - B)\cos 2\theta + 2C\sin 2\theta)(2C\cos 2\theta - (A - B)\sin 2\theta)^2} \end{aligned} \quad (35)$$

For  $T^\dagger[1] = [v_\mu]$ ,

$$\begin{aligned} \alpha_0 &= -\frac{(A - B)\cos 2\theta + 2C\sin 2\theta}{(A - B)} \\ \alpha_1 &= -\frac{\sigma^2 \sin 2\theta}{(A - B)C} \\ \alpha_2 &= -\frac{\sin 2\theta}{2(A - B)C} \end{aligned} \quad (36)$$

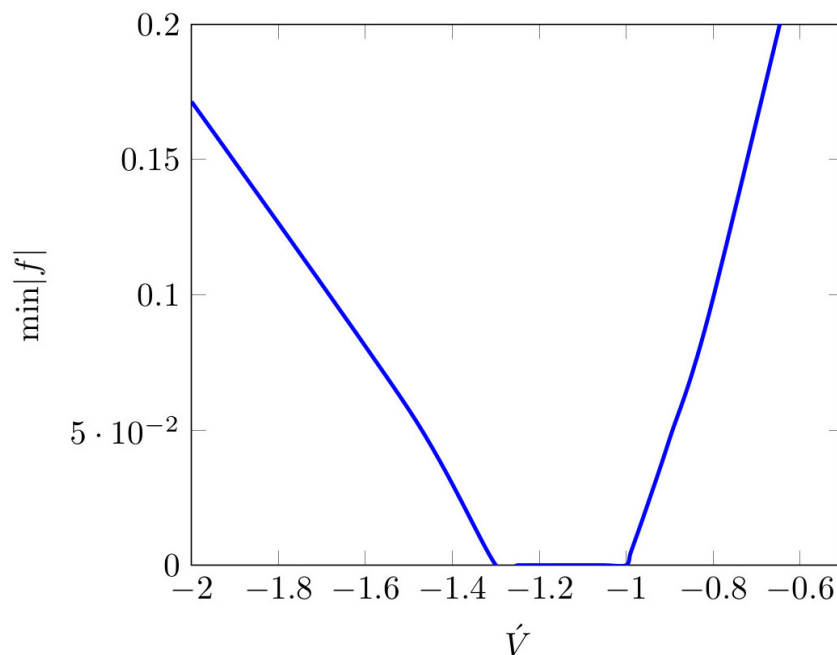
We further parameterize and scale the time  $t$  in units of  $E$  as  $V_0 = \hat{V} \frac{\Delta m_{21}^2}{2E}$ ,  $\sigma^2 = \hat{\sigma}^2 \frac{\Delta m_{21}^2}{2E}$ , and  $t = \hat{t} \frac{2E}{\Delta m_{21}^2}$ , then solve for  $f(\hat{V}, \hat{\sigma}^2) = 0$ . This renders  $\hat{V}$  and  $\hat{\sigma}^2$  to be dimensionless, and  $\frac{\sqrt{3}}{6}(X + y)\hat{t}$  is in radians. Furthermore, this also factors out  $\frac{\Delta m_{21}^2}{2E}$  from Equation (34), making the computations more precise. We also use known experimental estimates for solar neutrinos:  $\theta = 33.68^\circ \pm_{-0.70}^{+0.73}$  [36] for the computation. For each case, we substitute  $\theta$ , and the corresponding parameters  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$ , and plot the function  $f$  using the SageMath plot function to search for solutions for  $f = 0$ .

## 6. Results

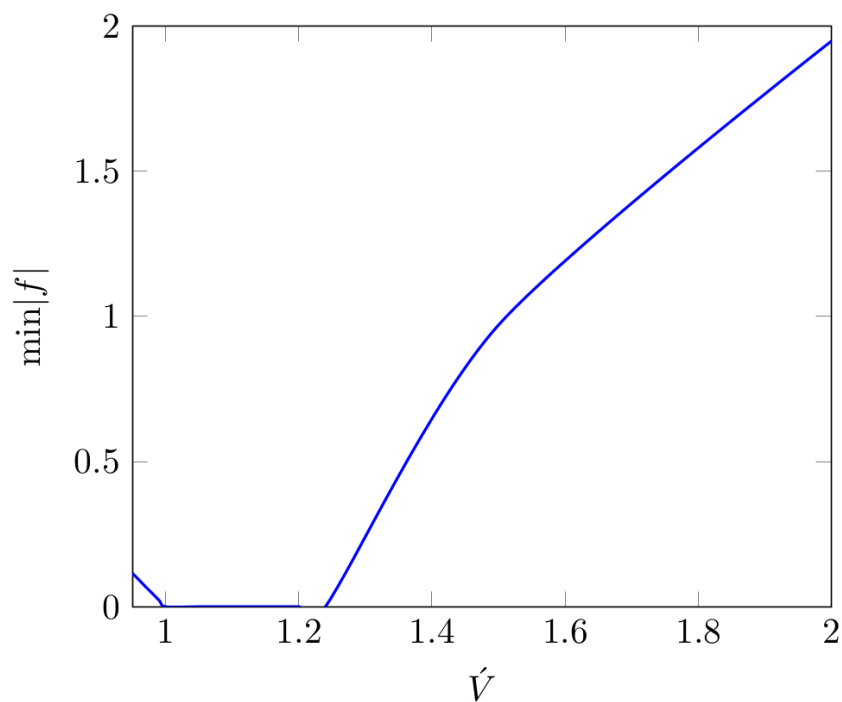
Initially, we obtained the range of values of  $\hat{V}$  such that  $f(\hat{V}, \hat{\sigma}^2) = 0$ , for  $\hat{\sigma}^2 \geq 0$ . Once the range of values of  $\hat{V}$  has been computed, we look for a corresponding set of values of  $\hat{\sigma}^2$  that led to  $f = 0$ . Figure 1 shows the minimum of the absolute value of  $f$  for values of  $\hat{V}$  for case (i), where  $T[v_e] = [1]$ . Figure 2 shows the minimum of the absolute value of  $f$  for values of  $\hat{V}$  for case (ii), where  $T^\dagger[1] = [v_\mu]$ . Both figures have been zoomed into the range where  $\min |f|$  nears 0 only.

For  $T[v_e] = [1]$ , Figure 1 shows that Equation (34) is satisfied for negative values of  $\hat{V}$ ,  $\approx -1.3 \leq \hat{V} \leq -1$ , that is  $\approx -1.3 \frac{\Delta m_{21}^2}{2E} \leq V_0 \leq -1 \frac{\Delta m_{21}^2}{2E}$ . This we reject as non-physical. For  $T^\dagger[1] = [v_\mu]$ , the solution lies with the range  $1 \leq \hat{V} \leq \approx 1.24$ , that is  $\frac{\Delta m_{21}^2}{2E} \leq V_0 \leq \approx 1.24 \frac{\Delta m_{21}^2}{2E}$ .

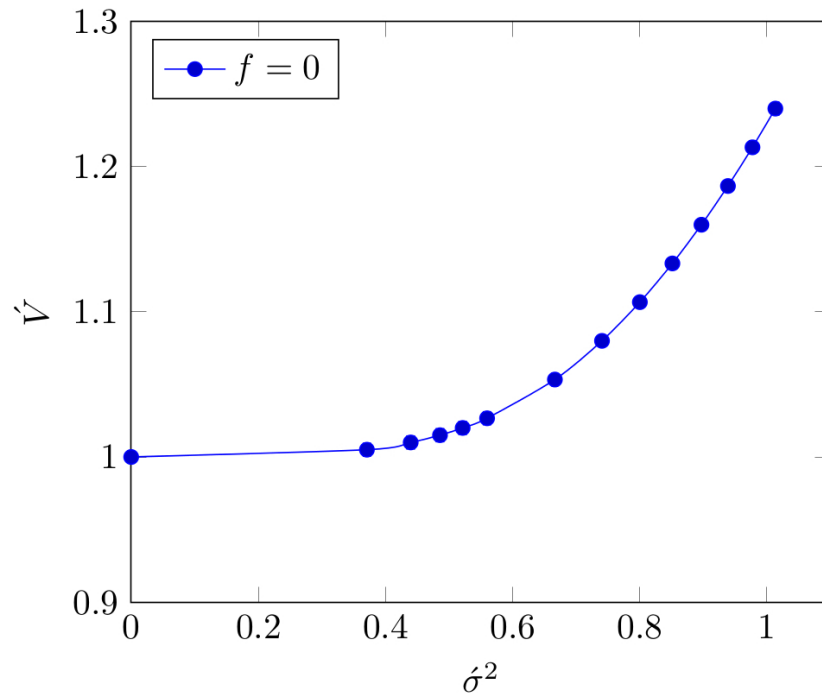
We proceed further to search for consistency in the second case only, that is  $T^\dagger[1] = [v_\mu]$ . Figure 3 displays the set of values for  $\hat{V}$  and  $\hat{\sigma}^2$  where  $f(V_0, \sigma^2) = 0$  for  $T^\dagger[1] = [v_\mu]$ . The figure is followed by Table 1 which lists the values used for plotting Figure 3, along with the corresponding values for  $\frac{\sqrt{3}}{6}(X + y)\hat{t}$ . The dots in Figure 3 are the exact data points that render  $f = 0$ . The curve through the points represents the continuity of the solution along the curve in the range on the figure.



**Figure 1.** Set of values of  $\dot{V}$  and  $\min |f|$  for  $T[v_e] = [1]$ . The figure plots the minimum of the absolute value of  $f$ , vertical axis, for  $\sigma^2 \geq 0$ , against each value of  $\dot{V}$ , horizontal axis. The figure has been zoomed in to show the range where  $\min |f|$  nears 0 only.



**Figure 2.** Set of values of  $\dot{V}$  and  $\min |f|$  for  $T^+ [1] = [v_\mu]$ . The figure plots the minimum of the absolute value of  $f$ , vertical axis, for  $\sigma^2 \geq 0$ , against each value of  $\dot{V}$ , horizontal axis. The figure has been zoomed in to show the range where  $\min |f|$  nears 0 only.



**Figure 3.** Plot of set of values of  $\dot{V}$  (vertical axis) and  $\sigma^2$  (horizontal axis) that satisfy Equation (34) for  $T^+[1] = [v_\mu]$ . The circles represent the data points in Table 1. The curve represents continuity of solution to Equation (34) for the range of values in Table 1.

**Table 1.** Data points for  $T^+[1] = [v_\mu]$ , where Equation (34) is satisfied, that is  $f(V_0, \sigma^2) = 0$ .

| $\dot{V}$         | $\sigma^2$                   | $\frac{\sqrt{3}}{6}(X+y)\dot{t}$ / radians |
|-------------------|------------------------------|--------------------------------------------|
| 1                 | 0                            | $\pi$                                      |
| $\frac{201}{200}$ | $0.3709^{+0.0055}_{-0.0055}$ | $1.1388\pi^{+0.0002\pi}_{-0.0002\pi}$      |
| $\frac{101}{100}$ | $0.4398^{+0.0058}_{-0.0059}$ | $1.1684\pi^{+0.0003\pi}_{-0.0004\pi}$      |
| $\frac{203}{200}$ | $0.4859^{+0.0058}_{-0.0059}$ | $1.1892\pi^{+0.0004\pi}_{-0.0005\pi}$      |
| $\frac{51}{50}$   | $0.5216^{+0.0057}_{-0.0058}$ | $1.2059\pi^{+0.0005\pi}_{-0.0006\pi}$      |
| $\frac{77}{75}$   | $0.5601^{+0.0054}_{-0.0057}$ | $1.2245\pi^{+0.0007\pi}_{-0.0008\pi}$      |
| $\frac{79}{75}$   | $0.6667^{+0.0041}_{-0.0045}$ | $1.2794\pi^{+0.0015\pi}_{-0.0016\pi}$      |
| $\frac{27}{25}$   | $0.7408^{+0.0027}_{-0.0031}$ | $1.3204\pi^{+0.0024\pi}_{-0.0025\pi}$      |
| $\frac{83}{75}$   | $0.8004^{+0.0013}_{-0.0018}$ | $1.3549\pi^{+0.0034\pi}_{-0.0034\pi}$      |
| $\frac{17}{15}$   | $0.8516^{+0.0000}_{-0.0004}$ | $1.3857\pi^{+0.0000\pi}_{-0.0046\pi}$      |
| $\frac{29}{25}$   | $0.8973^{+0.0010}_{-0.0015}$ | $1.4142\pi^{+0.0083\pi}_{-0.0076\pi}$      |
| $\frac{89}{75}$   | $0.9390^{+0.0024}_{-0.0029}$ | $1.4414\pi^{+0.0124\pi}_{-0.0110\pi}$      |
| $\frac{91}{75}$   | $0.9776^{+0.0039}_{-0.0044}$ | $1.4680\pi^{+0.0167\pi}_{-0.0148\pi}$      |
| $\frac{31}{25}$   | $1.0136^{+0.0054}_{-0.0024}$ | $1.4946\pi^{+0.0000\pi}_{-0.0165\pi}$      |

$\theta = 33.68^{+0.73}_{-0.70}$  [36], and time is scaled in units of  $E$ . Values for columns 2 and 3 computed through SageMath *find root* function. All values are approximated to 4 decimal places.

## 7. Discussion

The family of histories for this paper (15) is consistent if and only if the dynamics of the time evolution satisfy Equation (32). It is important to note that the dynamics can result in either  $T[v_e] = [1]$  or  $T^+[1] = [v_\mu]$ , but both cannot be satisfied simultaneously, as they represent mutually exclusive events in this formalism.

In the first case, similar to the results in [29], the family of histories would be *mathematically* consistent for negative values of  $\hat{V}$ , and consequently  $V_0$ , as shown in Figure 1. However, we reject this solution as non-physical.

In the second case, the family of histories is consistent for the set of values shown in Figure 3 and Table 1. The first set of values for consistency corresponds to the scenario where there is no continuous non-selective measurement of the ‘Mass’ Observable, i.e.,  $\sigma^2 = 0$ . In this case, the parameters reduce to those presented in [29], where consistency is achieved for the specific value of  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , such that  $Dt = n\pi$ , with  $n$  being odd. The introduction of the back-effect from the environment, mimicking continuous non-selective measurement of the ‘Mass’ Observable, leads to a broader set of values that achieve consistency, compared to the case without this *measurement*. Moreover, this back-effect does not introduce any dependence on the CP-violating Majorana Phase, meaning the consistency holds for both Majorana and Dirac neutrinos.

Let us now expand further on the results from Figure 3 and Table 1. For  $\theta = 33.68^\circ +0.73_{-0.70}$  [36], consistency is achieved when  $V_0$  lies within the range  $\frac{\Delta m_{21}^2}{2E} \leq V_0 \leq \approx 1.24 \frac{\Delta m_{21}^2}{2E}$ . The back-effect corresponds to a unique range for  $\sigma^2$ , specifically  $0 \leq \sigma^2 \leq \approx 1.0136 \frac{\Delta m_{21}^2}{2E}$ . The time intervals between events take corresponding values within the range  $\pi \frac{2E}{\Delta m_{21}^2} \leq \frac{\sqrt{3}}{6}(X+y)t \leq \approx 1.4946\pi \frac{2E}{\Delta m_{21}^2}$ . Since  $\frac{\sqrt{3}}{6}(X+y)t$  is  $2\pi$ -periodic, multiples of  $2\pi$  can be added to the range to obtain longer intervals between events if needed.

Let us now consider a specific set of values that lead to consistency, and refer to them as  $(V_{cr}, \sigma^2 cr, tcr)$ . If the dynamics of the time evolution are such that  $V_0 = V_{cr}$ ,  $\sigma^2 = \sigma^2 cr$ ,  $t_1 = tcr$ , and  $t_2 \neq t_1$ , then the family of histories is consistent, and  $T_1^\dagger[1] = [v_\mu]$ . In this case, the weights of histories  $M^1$  and  $M^2$  become 0, and they become dynamically impossible. Similarly, if the dynamics are such that  $V_0 = V_{cr}$ ,  $\sigma^2 = \sigma^2 cr$ ,  $t_2 = tcr$ , and  $t_1 \neq t_2$ , then the family of histories is consistent, and  $T_2[1] = [v_\mu]$ . This leads to the weights of histories  $M^2$  and  $M^3$  becoming 0, making them dynamically impossible. Finally, if  $V_0 = V_{cr}$ ,  $\sigma^2 = \sigma^2 cr$ , and  $t_1 = t_2 = tcr$ , the family of histories is consistent, with  $T_1^\dagger[1] = [v_\mu]$  and  $T_2[1] = [v_\mu]$ . In this case, histories  $M^1$ ,  $M^2$ , and  $M^3$  become dynamically impossible.

## 8. Conclusions

Neutrino oscillation remains a phenomenon studied in various disciplines and shows up in various interdisciplinary studies. One of those even goes beyond orthodox interpretations of Quantum Mechanics, namely Consistent Histories Formalism, where the time evolution is entirely probabilistic, and not just at *measurement*. Families of histories composed of quantum events where a qubit approximation of neutrino starting in an electron flavor ends up in one of the mass states after some evolution in time, and then further evolves in time and gets back to one of the flavor states in a consistent family of histories, however, only for a particular value of the interaction potential and time interval(s) between events [29].

Neutrino oscillation was also studied in the presence of an environment that mimics continuous non-selective measurement which leads to the suppression of the neutrino oscillations to an imperfect Zeno-like mechanism [17]. However, while this family of histories remains inconsistent for the majority of the range of values of the dynamics, this back-effect from continuous non-selective measurement that suppresses the oscillations leads to the expansion of parameters of the dynamics of the evolution, and consistency is achieved for a range of values instead of one particular value only without this measurement.

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## 10 Conclusions

Each article considered a set of different properties of a single neutrino in an initial state of that of electron neutrino flavor state and external conditions applied to this system. Qutrit and Qubit simplifications of the highly complex system of neutrino oscillations allows one to draw out its various dynamical properties in diverse models representing many disciplines of Physics.

Neutrino oscillations have already given so much to Physics. They were the first evidence of Physics beyond the Standard Model [C16]. This further led to the understanding of a non-correspondence between the mass and flavor states for neutrinos, and the qutrit and qubit dynamics for this phenomenon. Experiments have already computed bounds and constraints for most of the parameters that govern the dynamics of this phenomenon [A18].

The Mikheyev–Smirnov–Wolfenstein (MSW) effect showed that the oscillations for neutrinos propagating in matter differ from those propagating in vacuum, and provided the resonance condition for a maximal probability for change of flavor for neutrino. However, there are still open questions.

Recent experiments [B54,C3] have gone further to indicate a possible violating of Charge Conjugation and Parity (CP) symmetry for neutrinos, but the CP violation studied in the articles is still hypothetical, albeit the increasing evidence. Measurement of mass of neutrino is still not accessible, provided it is not technically impossible to begin with, the latter being one of the ideas explored in Article C, in as much as it being *mathematically consistent* for neutrino in a mass state being a *quantum event*. The Dirac or Majorana nature of neutrino is also one of the open questions.

While it might be an overstatement to state that all finite dimensional systems should be studied as open systems to shed further light on measurements and the properties of the system, this dissertation makes the case that to delve further into the properties of neutrinos, neutrino oscillations, and the open questions about them, one needs to study its properties in light of the impact of external conditions on them, and study it as an open system. One might also need to consider methods of investigation that go beyond the orthodox interpretation of quantum physics as well, as was done in articles C and D.

In light of the above, article A investigated neutrino oscillation coupled with an environment mimicking continuous non-selective measurement of the flavor and mass observables of the neutrino. Continuous measurements induced by a system's environment are sufficient to induce the quantum-to-classical transition, in which the trajectories of classical dynamics emerge as a result of quantum mechanics [A23], and thus make for a fascinating study.

The inclusion of normal matter or CP violation alone brings out changes to the interflavor probabilities, without even invoking a continuous non-selective measurement. Articles A and B both show that the presence of normal matter modifies the period of the oscillations for probability of transition of flavors, and increases the probability for change of flavor from the initial electron flavor. CP-violating phase did not bring about changes in periodicity, nor the crests of the transition probabilities. It damped the troughs of the oscillations of the transition probabilities, while the crests stayed the same.

The probability of transition from electron flavor to muon was increased significantly with the addition of continuous non-selective measurement of the flavor and mass observables, and even more so when hypothetical CP violation was allowed. For the two remaining transitions, electron to electron, and electron to tau, continuous non-selective measurements led to the damping of oscillations, which was stronger for the systems coupled through the mass observable to the environment.

Continuous non-selective measurements also damped the oscillations of the autocorrelation of the flavor observable in the presence of normal matter and CP violation, and for times shorter than the equilibrium time, autocorrelation was indistinguishable for whether the environment coupled via flavor or mass observable.

Another external condition that can be applied to neutrino oscillation is that of Decoherence. Even a simple Markovian model for Decoherence, further simplified with including only diagonal decoherence, impacted the probabilities as well. Decoherence also did not influence the periodicity of neutrino oscillations. It damped the crests of the oscillations of the transition probabilities, while the troughs stayed the same.

Another property worth studying for neutrino oscillations is Geometric Quantum Speed Limit  $\nu_{QSL}$ . Geometric QSL represents the speed of change of

*distance* between states [B28,B29], in the sense of Fidelity, and thus in the absence of normal matter,  $\nu_{QSL}$  shows discontinuities around the windows where the neutrino is at its initial state, or returns to its initial state, of electron neutrino flavor. The presence of matter, however, not only changes the periodicity of  $\nu_{QSL}$ , it also largely removed these discontinuities and replaced them with sharp but continuous peaks.

The impact of diagonal Decoherence on Geometric QSL was observed too. It also worked towards removing the discontinuities in  $\nu_{QSL}$  that are visible for Decoherence free scenario, just like there are discontinuities visible in absence of matter. Significantly, for sufficiently large values quantifying the decoherence,  $K$ ,  $\nu_{QSL}$  becomes a monotonically decreasing function of time, signifying a move towards *classical system like equilibrium*.

Geometric QSL shines brightest for the case of CP-violating phase. In the vicinity of the local maxima of  $P_0$ , the overlap probability, that is when the neutrino returns close to its initial state,  $P_0$  and  $\nu_{QSL}$  are indistinguishable with respect to change in the values of CP-violating phase. Outside these windows, non-vanishing CP-violating phase enhanced the other peaks of  $\nu_{QSL}$ , those that correspond with the neutrino being in the local minima of the overlap probability  $P_0$ , signifying the neutrino being further possible from the initial state, in the sense of Fidelity, for that particular oscillation.

Thus, Geometric QSL can serve as a tool that quantifies CP violation provided that the time windows where  $\nu_{QSL}$  is identical for all values of  $\delta$  is accounted for.

When it comes to illuminating the accessibility of mass of neutrinos, it is worth considering models and approaches to quantum theory that go beyond its orthodox interpretation. *Consistent Histories* or *Decoherent Histories* formalism is one of them.

Without invoking *measurement*, Consistent Histories Approach allows the assignment of probabilities to a family of histories of quantum events of a system provided that the family is consistent. The dynamics of the system, and the choice of events to design the family of histories are of utmost importance here.

Neutrino was taken as a qubit approximation for this. A choice that allows analytical solution to this problem while still showing dynamical properties. The simplest non-trivial history is a three-time history, and that was de-

signed for 2-level neutrino. The beginning and ending points of the histories are "events" of a neutrino in a flavor state. However, as the inter-mediate event of the three-time history, there is a neutrino in a pure mass. state.

The family of events proposed is inconsistent when a neutrino propagated in vacuum. That is to say that, under this approach, one cannot credibly assign probability to the events where neutrino is in a pure mass state between being in two pure flavor states while propagating in vacuum. However, the family of histories became consistent when a neutrino propagated in the presence of normal matter, provided that an interaction potential,  $V_0$ , and the time interval(s) between the events took particularly exact values.

There are two intervals between the three events in each history,  $t_1$  between initial pure flavor state to pure mass state, and  $t_2$  between the pure mass state to pure flavor state. Whether  $t_1 = t_2$ , and the choice of  $D$ , which is the difference between the two eigenvalues of the system, significantly affects probabilities (weights) of the histories.

If  $t_1 \neq t_2$  and  $n$  is an odd number such that we have  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t_1}$ , and  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , the family is consistent. However, histories  $M_1$  and  $M_2$  then have zero weights, and these events do not occur at all. The only events that have non-vanishing probabilities from this family are:

$$M^0 : [v_\mu] \odot [I] \odot [I]$$

$$M^3 : [v_e] \odot [2] \odot [v_\mu]$$

$$M^4 : [v_e] \odot [2] \odot [v_e]$$

An electron neutrino will always be in the 2nd mass state after  $t_1$ , but can be found in either flavor state if  $t_2 D \neq n\pi$ , where  $n$  is an odd number.

If  $t_1 \neq t_2$  and  $n$  is an odd number such that we have  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t_2}$ , and  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , the family is also consistent. However, histories  $M_2$  and  $M_3$  then have zero weights, and the events that have non-vanishing probabilities from this family are:

$$M^0 : [v_\mu] \odot [I] \odot [I]$$

$$M^1 : [v_e] \odot [1] \odot [v_\mu]$$

$$M^4 : [v_e] \odot [2] \odot [v_e].$$

An electron neutrino can be in 1st or 2nd mass state if  $t_1 D \neq n\pi$ , where  $n$  is an odd number. However, they correspond with the third event being muon neutrino and electron neutrino respectively after  $t_2$ . The flavor at the third event can infer the pure mass state at the intermediate one.

When  $t_1 = t_2 = t$ ,  $D = 2\sin\theta \frac{\Delta m_{21}^2}{2E} = \frac{n\pi}{t}$ , and  $V_0 = \frac{\Delta m_{21}^2}{2E}$ , such that  $n$  is an odd number, the family of histories is consistent. However, histories  $M_1$ ,  $M_2$  and  $M_3$  then have zero weights, and the events that have non-vanishing probabilities from this family are:

$$\begin{aligned} M^0 &: [v_\mu] \odot [I] \odot [I] \\ M^4 &: [v_e] \odot [2] \odot [v_e]. \end{aligned}$$

A neutrino in initial electron flavor state will be found in electron flavor state once  $t_1 + t_2$  has elapsed, a conditional probability of 100%, and we can also infer that the intermediate mass state after  $t_1$  was the 2nd one, and by symmetry, same calculations will show that given the initial flavor is that of a muon neutrino, the *conditional survival possibility* will also be 100%, and the intermediate mass state after  $t_1$  will be the 1st one.

Article D then takes the same family of histories but changes the dynamics of the system by invoking a continuous, non-selective measurement of one of the observables from Article A, hypothetical mass state measurement. This *measurement* like mechanism introduces a back-effect on the system of neutrino propagating in normal matter, quantified as  $\sigma^2$ , equation 6 of the article, changing the dynamics from unitary to Markovian, resulting in a Quantum Master Equation, which is equation 5 of the Article.

This family of histories is still consistent even in the presence of back-effect from environment mimicking a continuous non-selective measurement of *mass observable*, and just like the case in Article C without this measurement, it is inconsistent for the majority of the range of values of the dynamics. However, without this measurement, it is consistent only for the one set of values of the parameters shown in Article, with the inclusion of this *measurement*, expands the parameters of the dynamics of the evolution that lead to consistency, and it is achieved for a range of set of values of the parameters,  $\frac{\Delta m_{21}^2}{2E} \leq V_0 \leq \approx 1.24 \frac{\Delta m_{21}^2}{2E}$ ,  $0 \leq \sigma^2 \leq \approx 1.0136 \frac{\Delta m_{21}^2}{2E}$ , and  $\pi \frac{2E}{\Delta m_{21}^2} \leq \frac{\sqrt{3}}{6}(X + y)t \leq \approx 1.4946\pi \frac{2E}{\Delta m_{21}^2}$ . Since  $\frac{\sqrt{3}}{6}(X + y)t$  is  $2\pi$ -periodic, multiples of  $2\pi$  can be added to the range to obtain longer intervals between events.

The qualitative nature of the results from Article C still stand for this open system interacting with environment. The choice of  $t_1$  and  $t_2$  corresponding with set of values for  $V_0$  and  $\sigma^2$  leads to the same set of weights for individual histories and results as discussed above. However, this interaction with the environment, which in article A suppresses the neutrino oscillations, expands the set of parameters that achieve consistency, making it relatively likelier to have the *quantum event* of neutrino being in its pure mass state than that when it is a closed system, but to an extent only.

As per [C38], estimates for solar neutrinos put  $\Delta m_{21}^2 = 1.4906 \times 10^{-60} m_P$ , ( $8 \times 10^{-5} eV^2$ ). More recent estimates considered in Article D [D36] put it as  $\Delta m_{21}^2 = 7.49_{-0.19}^{+0.19} \times 10^{-5} eV^2$ . The magnitudes of  $V_0$  and  $t$ , and  $\sigma^2$  when measurement from environment is included, that provide consistency are extremely small for even a small value of  $E$ . The values of  $E$  that would be expected in a real system of neutrinos, these parameters would be even smaller. The more accurately the environment measures the observable, more information is gained, the larger  $\sigma$  is [A25]. If the information gained from the measurement is such that  $\sigma^2$  is greater than the range from article D, which is quite small to begin with, the family of histories is inconsistent. This provides additional evidence for the neutrino mass states being hardly accessible, even from a fundamental perspective, and not only a technical viewpoint.

The consistency of such a family of histories is not affected by the CP-violating Majorana phase. That being stated, the results still stand for both Majorana or Dirac neutrinos and would be unchanged after the CP problem is fully resolved.

## 11 Author Contributions

All four articles that make up the dissertation were a Collaboration between the author and their supervisor.

In particular, the contribution for Article A was:

Conceptualization, F.K. and J.D.; methodology, J.D.; software, F.K. and J.D.; formal analysis, F.K. and J.D.; writing—original draft preparation, J.D.; writing—review and editing, F.K. and J.D.

The contribution for Article B was:

Conceptualization, F.K. and J.D.; methodology, J.D.; software, F.K. and J.D.; formal analysis, F.K. and J.D.; writing—original draft preparation, J.D.; writing—review and editing, F.K. and J.D.

The contribution for Article C was:

Calculation, F.K.; Investigation, F.K. and J.D.; Writing—Review and Editing, F.K. and J.D.

The contribution for Article D was:

Calculation, F.K.; Investigation, F.K. and J.D.; Writing—Review and Editing, F.K. and J.D.