Abstract

If $\sum_n x_n$ is an unconditionally convergent series in a Banach space, then the set

$$A(x_n) := \left\{ \sum_{n \in A} x_n \colon A \subseteq \mathbb{N} \right\}$$

is well defined and we call it the *set of subsums* or the *achievement set* of sequence (x_n) (or series $\sum_n x_n$). The set of subsums of an absolutely convergent series on the real line has one of the following forms: a finite set, a finite union of closed intervals, a Cantor set of an M-Cantorval. In turn, an algebraic sum of two homogeneous Cantor sets has one of the following forms: a closed interval, a Cantor set, an L-Cantorval, an R-Cantorval or an M-Cantorval. These facts serve as motivation to study Cantorvals. Cantorvals are compact regularly closed subsets of the line, which combine the properties of the Cantor set and the interval. In chapter one I report the result of my co-authored publication:

[I] W. Bielas, M. Kula, Sz. Plewik. *On compact subsets of the reals.* Topology Appl. 346, Paper No. 108854 (2024), 10 p.

We prove the characterization of compact metric spaces that can be topologically embedded into the line. We use it to prove equivalent conditions for the space to be homeomorphic with M-Cantorval or L-Cantorval. Using the properties of countable linearly ordered spaces, we give proofs that any two M-Cantorvals are homeomorphic and any two special L-Cantorvals are homeomorphic. Using Mazurkiewicz–Sierpiński theorem on the number of non-homeomorphic subspaces of rationals, we prove that there exist continuum many non-homeomorphic L-Cantorvals.

One of the fundamental questions concerning sets of subsums are criteria for determining the topological form of the set of subsums of a particular series. On the real line this problem is reduced to series of positive terms in descending order. The so-called Kakeya inequalities, i.e. relationships between the terms and remainders of the series, allow to decide the topological form of the set of subsum, but not in all cases. Known criteria do not always provide an answer as to whether a given set of subsums is a Cantor set or an M-Cantorval. In chapter two I present thus far unpublished result of joint work with Szymon Głąb. After discussing known properties of sets of subsums in

general, we deal with the self-similar sets

$$E(\Sigma, q) := \left\{ \sum_{i=1}^{\infty} \sigma_i q^i \colon (\sigma_i) \in \Sigma^{\mathbb{N}} \right\},$$

where $\Sigma \subseteq \mathbb{R}$ is a finite set and $q \in (0,1)$. This notion is a generalization of the set of subsums of multigeometric sequence, which is the sequence of the form

$$(p_1, \ldots, p_n; q) := (p_1 q, \ldots, p_n q, p_1 q^2, \ldots, p_n q^2, \ldots),$$

where $p_1, \ldots, p_n$ are real numbers and $q \in (0,1)$. Our joint result is a new characterization, when the set $E\left(\Sigma, \frac{1}{|\Sigma|}\right)$ contains an interval. Under some assumptions it can be used to determine the topological form of the set of subsums of multigeometric series.

Chapter three concerns sets of subsums on the plane $\mathbb{R}^2$. I present the results of the publication:

[II] M. Kula, P. Nowakowski. *Achievement sets of series in* $\mathbb{R}^2$. Results Math. 79 no. 6, Paper No. 221 (2024), 24 p.

We give examples of sets of subsums of various topological form, in particular we deal with different possible connected components. We prove the existence of a series whose set of subsums has a cut that is a given set of P-sums. We use this result to construct a counterexample to the conjecture that any set of subsums in $\mathbb{R}^2$ up to homeomorphism is a finite union of products of sets of subsums in $\mathbb{R}$.

The center of distances is an important tool in studying sets of subsums. For example, this notion is used in proofs that some set cannot be represented as a set of subsums of any series. In chapter four I give a positive answer to Małgorzata Filipczak's question, whether any set $A \subseteq [0, \infty)$ with $0 \in A$ is a center of distances of some subset of the real line. Then I strengthen this result by showing that such subset can be in addition chosen to be a Bernstein set. These results are published in the article:

[III] M. Kula. *Center of distances and Bernstein sets.* Real Anal. Exchange, Advance Publication 1 – 6 (2025), `https://doi.org/10.14321/realanalexch.1739330962`