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Applications of topos theory to quantum physics

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Declaration of Authorship

I hereby declare that the thesis entitled "Applications of topos theory to quantum physics", submitted for the award of the degree of Doctor of Philosophy to University of Silesia, is a record of bonafide work carried out by me under the supervision of dr hab. Jerzy Król from University of Information Technology and Management in Rzeszów. I further declare that the work reported in this thesis has not been submitted and will not be submitted, either in part or in full, for the award of any other degree or diploma in this institute or any other institute or university.

Signed by the Candidate:

Date: October 27, 2023

UNIVERSITY OF SILESIA

Abstract

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Applications of topos theory to quantum physics

by Krzysztof BIELAS

This thesis explores various applications of topoi in the realm of quantum physics. In particular, the work employs a method of variable mathematical foundations, wherein the conventional topos **Set** is consequently substituted with sheaf topoi $Sh(B)$ (derived from Boolean-valued models V^B built upon Boolean algebras of projections on a Hilbert space). On occasion, the Basel topos $\mathcal{B}$ is also utilized. It is argued that the approach may shed new light on the cosmological constant problem by altering the structure of real line and, consequently, the smooth structure of spacetime. Furthermore, the possible connection with exotic smooth structures on $\mathbb{R}^4$ is discussed. Eventually, the problem of randomness of quantum mechanics is addressed and it is demonstrated that a quantum system described by the infinite-dimensional Hilbert space formally exhibits a stronger notion of randomness.

Keywords— quantum mechanics, topoi, Boolean-valued models, exotic smoothness, randomness

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Contents

Chapter 1

Introduction

1.1 History and motivations

The state of physics appeared to be well-established at the turn of 19th and 20th centuries, as the matter and radiation got the successful description through Newton's laws of motion, thermodynamics, and Maxwell's equations. The challenging aspects were not perceived as major obstacles in the general picture. In fact, fields like electromagnetism or thermodynamics were regarded as mature and reliable theories that would need only some subtle refinements to achieve completeness. The tone of a discussion of the time could be characterized by a quite symptomatic claim, attributed to Jolly [124]: "In this field, almost everything is already discovered, and all that remains is to fill a few holes."

Very easygoing as it was, it turned out to be too optimistic. In fact, there were few who could predict huge changes that would shake the foundations of our description of Nature. Surprisingly, it was Lord Kelvin who, despite many misattributions, foresaw that the problems of aether and equipartition theorem (referred to as "clouds" by Kelvin in [99]) would cause a revolution; as we know, these issues found resolutions in the equations of special relativity and quantum mechanics, respectively. Indeed, starting with seemingly innocent assumptions such as Planck's hypothesis of radiation emitted in quanta of energy $h\nu$ for a given frequency ν one can derive the formula for the radiated energy density of a black-body as [149]

$$u(\nu, T) = \frac{a\nu^3}{c^3} \frac{1}{\exp\left(\frac{b\nu}{T}\right) - 1}$$

which circumvented the Rayleigh–Jeans ultraviolet catastrophe and established a link with Wien's displacement law.

Considering atomic spectra, two key approaches evolved to adress the observational discrepancies, namely Heisenberg's matrix mechanics and Schrödinger's wave mechanics. The former was derived from the relations

$$\nu = cR\left(\frac{1}{n_a} - \frac{1}{n_b}\right) \tag{1.1}$$

where R — Rydberg's constant; as $n_a, n_b \in \mathbb{N}$, discrete nature becomes apparent here. It is worth emphasizing that (1.1) was established via experiments, without any theoretical underpinning. Then, the remarkable insight of Heisenberg was the translation of (1.1) into the language of matrices [79], what initialized the matrix approach to quantum mechanics introducing the fundamental commutation relations

$$Q_i P_j - P_j Q_i = \mathrm{i}\hbar\delta_{ij}, \tag{1.2}$$

which finally led to the Heisenberg's uncertainty principle

$$\Delta Q \Delta P \geq \frac{\hbar}{2}.$$

A somewhat distinct path was chosen by Schrödinger who, inspired by de Broglie, suggested that for an orbiting electron, one could associate a complex-valued wave function $\psi(x,t)$, that obeys a key relation

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V\psi(x,t), \tag{1.3}$$

now known as the Schrödinger's equation. Then, provided the interpretation for Q, P operators

$$\big(Q(\psi)\big)(x,t) = x \cdot \psi(x,t), \quad \big(P(\psi)\big)(x,t) = -\mathrm{i}\frac{\partial}{\partial x}\psi(x,t),$$

one proves these satisfy (1.2).

Initially, in spite of successes in clarifying certain aspects of the small-scale physics, these approaches appeared to be mutually exclusive due to differences in their mathematical formulations. It was von Neumann [142] who uncovered a unifying framework, namely the Hilbert space structure, lurking behind both concepts (we will come back to this topic in Chapter 2). In particular, it became apparent that the infinite matrices in (1.2) represented in fact operators acting on the space $\ell^2(\mathbb{N})$, while wave functions in (1.3) were the elements of $L^2(\mathbb{R}^3)$, and an isomorphism (unitary equivalence) between these spaces, recognized earlier by Dirac, provided the demanded equivalence. However, this joint framework of Hilbert spaces made the distinction between quantum and classical physics even more severe, as it made clear that quantum mechanics operated on entirely distinct claims, regarding what can be deemed "real".

So far we have briefly discussed the profound changes brought to life to understand better e.g. how particles and waves behave and interact. Significantly, the revolution was not limited to the characteristics of matter and fields, but it was extended to the stage they operate on, i.e. to space and time. Here, the first groundbreaking concept proposed by Einstein was the theory of special relativity, merging a so-far intuitive, global space and time notions into a complex and dynamical object of a spacetime. The theory introduced guiding principles such as the constancy of the speed of light and the invariance with respect to the choice of inertial frame of reference. Consequently, classical Galilean transformations were replaced by Lorentz transformations and the new invariant quantity was the spacetime interval

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2,$$

instead of the plain length element $\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, invariant under Galilean transformations. Therefore, the concept of an aether became obsolete and, starting with Michelson–Morley experiment, numerous observational discrepancies have been resolved. Several years later, special relativity was successfully merged with quantum mechanics, giving rise to quantum field theory (QFT).

The subsequent remarkable concept, once more initiated by Einstein, was the theory of general relativity. This time, the invariance (called general covariance) was enriched to include also non-inertial frames of reference, which led to the fundamental observation that gravity and acceleration are locally indistinguishable. Energy-mass distribution is then directly related to spacetime curvature and both phenomena can be covered by Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.4}$$

Provocatively, it is safe to end this story here. Of course, this does not imply that no progress has been made since then. There have been numerous developments such as the formulation of QFT as a Yang–Mills theory, the Standard Model, advancements in cosmology, nanophysics etc. However, one could argue that since the emergence of quantum mechanics and the "relativity breakthrough", no another revolution has arisen that would shake the foundations of physics as significantly as the ones that occurred in the beginning of 20th century.

This brings us to the central issue concerning contemporary physics: quantum gravity, which aims at a desired marriage of quantum mechanics and general relativity, essential to make coherent predictions in both micro- and macroscales. Leaving details to the subsequent section, we stress here the main difference with the matter discussed above: while the rise of quantum mechanics and (special) relativity was fully connected with the necessity of explaining observed phenomena, as we had no tools describing the world of microscales and high energies, nowadays we have separate theories that describe the world around us extremely well, at least in their scopes of applicability. Indeed, quantum mechanics, extended by QFT, enjoys predictions of enormous precision e.g. in particle colliders; similarly, general relativity describes well phenomena such as large-scale structures in the Universe, gravitational waves etc. Despite these achievements, the major theories significantly diverge with respect to the mathematics they employ (we discuss this topic further in Section 1.2). Not only do they differ given theoretical background, but do not agree on the level of the ontological principles as well, since quantum theory undermines objectivity of quantities treated within general relativity as real and meaningful [88]. Thus, not underestimating experimental side, greater emphasis is placed on the efforts toward explaining the origin of classical and quantum discrepancy, ultimately merging these descriptions into a unified picture.

Remark 1. Ironically, the list of contemporary unsolved problems in both experimental and theoretical physics may appear not only longer, but also more formidable than the one encompassing early 20th century topics. Today, this list contains the puzzles such as dark matter, high-temperature superconductors, masses of elementary particles or the value of cosmological constant, to mention a few (naturally, these problems have been present in the past, although not always recognized), see e.g. [69]. Now, bearing in mind the consequences the problems of 19th century physics have caused, one should rather prepare for even more dramatic reformulations that our contemporary questions may provoke.

All the above reflections on the present state of fundamental physics questions, together with lessons taken in the past, suggest that it might be beneficial to occasionally change our course and open the door to some fresh ideas. Obviously, in doing so one has to avoid the trap of ridiculous concepts that promise miraculous solutions to all the puzzles. Although physics, like the rest of contemporary science, includes some mechanisms that dismiss questionable approaches (i.e. an experimental verification), the field of mathematical physics is more involved. Here, direct predictions are often reproduced, and the emphasis is put rather on the clarification of observed relationships or highlighting previously hidden structures. Nevertheless, just as in the case of negating the spacetime as a global time and space product, non-Euclidean geometry in general relativity, functional analysis in quantum mechanics or local/global symmetries in gauge theory, we claim that problems arising on the classical-quantum boundary might demand even more unconventional departure from existing theoretical frameworks. One such departure we stand for is to relax the assumption of unique "mathematical background" used in various physical contexts. In particular, it boils down to the usage of "the same mathematics": identical rules for logic, objects and structures, despite significant differences between the theories employing them. In other words, it is by default to treat notions such as the law of excluded middle or real numbers as absolute and invariant, from cosmological scales down to subatomic particles. We advocate here a substantial generalization that comes from topos theory. Here, both model theory and category theory give formal tools to study the implications of such an approach.

Throughout the thesis, our focus remains on the exploration of a setup that appears promising but may seem overly oversimplified at first glance. More precisely, we examine a quantum system $\mathcal{S}$ in conjunction with the surrounding spacetime M; Although these entities are typically treated as independent and distant, it is essential to recognize their interplay at the formal level of mathematical structures associated with $\mathcal{S}$ and M. By "oversimplified", we mean an approach, that relies on minimal assumptions concerning $\mathcal{S}$ and M, namely:

- the system $\mathcal{S}$ is to be described by a standard quantum theory in the language of operator algebras,
- M is a smooth manifold and the topology of M is the standard Euclidean $\mathbb{R}^n$,
- the space of states of $\mathcal{S}$ is assumed to be infinite-dimensional, as it is a necessary condition for canonical commutation relations between position and momentum to be represented,
- the description of $\mathcal{S}$ and M is not limited to a single mathematical universe (e.g. neither to a single category or a single ZFC model) in principle.

All the above points are discussed in Chapter 4. Based on them, we will demonstrate in Chapter 3 an unexpected connection between the mathematical foundations of $\mathcal{S}$ and M. Specifically, every quantum system $\mathcal{S}$ equipped with position and momentum observables gives rise to an intricate logical structure comprising yes-no propositions related to $\mathcal{S}$. While this infinite structure does not align with classical logic, it can be covered or "approximated" by classical contexts. Surprisingly, each of these contexts serves as a seed to entire mathematical universe, family of which is carried by $\mathcal{S}$. These universes, i.e. respective ZFC models, give then the opportunity to approach the local, smooth structure of M through such logical contexts. Eventually, it appears that the smoothness of M cannot be the standard one.

Referring to unresolved questions, we consider a specific case of the cosmological constant (CC) problem, related closely to our comprehension of physical vacuum within classical gravity and quantum field theory. The CC problem may be regarded as one of the facets of quantum gravity questions, as it lies at the intersection of these regions via Einstein equations: although (1.4) describe the interplay of geometry and matter on a large scale, one argues that the term $\Lambda g_{\mu\nu}$ encompasses vacuum energy (among others), and it is quite straightforward to show that zero-point fluctuations of quantum fields is a major contribution here. However, the estimated magnitude of this contribution deviates from the experimental value significantly; depending on several technicalities, the discrepancy ranges from 55 to 120 orders of magnitude. It is thus not surprising it has been referred to as "the worst theoretical prediction in the history of physics" [84]. Remarkably, it appears that the approach presented in this work may shed some light onto the resolution of the CC conundrum. Firstly, it nullifies zero-point energy of quantum fields through set-theoretic model extension driven by forcing procedure. Secondly, it shows smooth structure of expanding spacetime to be an exotic one and finally provides hints toward the realistic value of CC. Next we turn to a categorical perspective on the atlas of smooth spacetime M, indexed by Boolean subalgebras of the lattice of projections $\mathcal{L}(\mathcal{H})$. Furthermore, approaching spacetime smoothness through the local ZFC models gives rise to another local modification of M: here, transition functions are locally modified through the special type of a smooth topos — a Basel topos, which again brings us to exotic smoothness. Finally we discuss how formal tools used so far relate to the quantum-mechanical randomness and its interpretation. In particular, we show that the recent results of Landsman [118] may be extended to support even stronger notion of randomness realized by quantum mechanics in infinite dimensions. It leaves also an open question of whether it will be experimentally accessible to verify and use such strong randomness at our disposal.

As title of the thesis suggests, there is a common thread in the above. Indeed, we discuss several alterations to the basic mathematical framework — a supposedly unique ZFC model — and proceed with replacing it locally by various topoi. The term "local" refers here either to:

- commutative contexts represented by Boolean subalgebras of $\mathcal{L}(\mathcal{H})$,
- open neighbourhoods in a smooth spacetime atlas.

The former typically gives rise to Boolean topoi, i.e. Boolean-valued models, while the latter introduces Basel topos, as mentioned above.

Since topics discussed in this work might be somewhat uncommon for both mathematicians and physicists, we have decided to provide a rather lengthy introduction to the tools used. On the one hand we try to be not too encyclopedic (a more detailed background with definitions and lemmas is provided in Appendices), on the other hand it should allow to read the Chapters 3, 4, 5, 6 without much struggle.

1.2 Classical and quantum

One of the major themes going through the thesis is the vast difference between classical and quantum physics. In fact, their disagreement can be seen on two levels: experimental and theoretical. These are related naturally, as the results of measurements ideally either validate theoretical constructs (e.g. Young's double-slit experiment) or give an impulse to the modifications of the theory (e.g. discrete atomic spectra). Thus, one may elaborate on the quantum-classical distinction either from experimental or theoretical standpoint.

What introduces the perspective of classical physics most effectively is perhaps the concept of an objective reality, i.e. the existence of parameters (real-valued functions) that characterize the system completely and independently of external observers. Put differently, the influence of apparatus on the measured object can always be made negligible, as long as the measurement concerns classical, macroscopic regime (although one could argue that classical chaotic systems do not fall into this category, due to their sensitivity to initial conditions, see also Chapter 6). This boils down to characterizing a state, encompassing all available information about a classical system as a point $x \in X$ where X is a certain phase space (manifold, measure space etc.). Then, above parameters become just (smooth, measurable) real-valued functions $\{f \mid f : X \to \mathbb{R}\}$ interpreted as physical quantities, such as energy, momentum etc.

Conversely, quantum mechanics is inherently connected with certain degree of unpredictability, at least in the general setting. A classic example could be the Stern–Gerlach experiment, where a set of polarizers not only reveals an inherently probabilistic nature of electron's spin, but shows also an inevitable influence that a measurement makes on the considered system. It may seem less surprising, though, as soon as one becomes acquainted with the mathematical formalism of a quantum theory. Here, the space of states takes on the form of a Hilbert space $\mathcal{H}$, and physical quantities are represented as self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$ acting on that space. Therefore, the quantum non-determinism of a measurement results is rooted in the fact that results are not represented by values of operators themselves, but rather as eigenvalues with eigenvectors assigned. Nevertheless, despite all the differences, one proceeds in a manner similar to the classical case. We elaborate on these differences and focus on the mathematical aspects of quantum mechanics in the Chapter 2.

1.3 (Real) numbers in physics

As mentioned earlier, one of the elements that sets classical and quantum physics apart, lies in the inherent nature of numbers that naturally emerge in each theory. Namely, while the continuity of real numbers seems to convey our experience of macroscopic world, the quantumness of microworld is inevitably connected to the discreteness of natural numbers (hence the difference lies mainly in the cardinality of the sets of experiment outcomes that are in principle available). This observation leads to a fundamental question: Do these distinct "natures" of various number systems have some irreducible physical significance, or do they merely reflect the universal principle of effective theories.

To play devil's advocate, one could naturally consider a finite precision of every measurement and claim that no such "global" property of a number system could affect the result of an experiment. After all, one may argue that any device's precision is fundamentally limited and does not allow to exactly pinpoint a physical quantity. In other words, the way a measurement apparatus is operated in principle boils down to handling rational numbers with a finite decimal expansion, and there is no possibility to tell apart e.g. the numbers $\sqrt{2}$ and 1.414213, if the given precision is 0.001 (in certain unit). In fact, even experiments designed to distinguish e.g. the rationals and irrationals cannot provide a sharp boundary. We refer here to the series of experiments (see e.g. [154]) based on the work of Hofstadter [85], where the (ir)rationality of a magnetic flux' value ϕ through the lattice controls the structure of energy bands of electrons on the lattice. In short, for rational ϕ, energy spectrum consists of the union of intervals, while for irrational ϕ one can prove that the spectrum becomes the Cantor set, a famous fractal-like object (see also Section 6.2 for more information on this peculiar set). Nevertheless it can be proved that, given a finite precision, one again cannot distinguish between a Cantor set and a particular interval-like spectra within a certain length scale, cf. [157].

Remark 2. Idealization is a prevalent practice in physics research, and one illustrative example can be found in statistical physics. In this context it is necessary to invoke the thermodynamic limit (both number of particles and volume go to infinity) in order to establish phase transitions. The reason is that it is impossible to obtain a singularity (non-analytic behaviour of a thermodynamic potential) in a finite systems. As phase transitions can be easily seen in macroscopic, $\sim 10^{23}$-particle systems, a thermodynamic limit should be clearly not necessary, yet it seems to be "forced" by the way we formulate the theory.

To complicate things even more, not all number systems are "the same" all the way through mathematical structures. In particular, real numbers appear to be highly dependent on the mathematical universe they inhabit. We explore this topic in Chapter 2 and apply it extensively in Chapters 4, 5, 6.

1.4 List of publications

This thesis is based mainly on the following published, co-authored material:

1. Chapter 4: [108] J. Król et al. "From Quantum to Cosmological Regime. The Role of Forcing and Exotic 4-Smoothness". In: *Universe* 3(2), 31 (2017), pp. 1–11. DOI: https://doi.org/10.3390/universe3020031

2. Chapter 5:

   - [30] K. Bielas, J. Król, and T. Asselmeyer-Maluga. "Building the Cosmological Models vs. Different Models of Set Theory". In: *Acta Physica Polonica B* 46 (Nov. 2015), p. 2369. DOI: 10.5506/APhysPolB.46.2369
   - [29] K. Bielas and J. Król. "From Quantum-Mechanical Lattice of Projections to Smooth Structure of $\mathbb{R}^4$". In: *Category Theory in Physics, Mathematics, and Philosophy*. Ed. by M. Kuś and B. Skowron. Cham: Springer International Publishing, 2019, pp. 83–93. ISBN: 978-3-030-30896-4

3. Chapter 6: [111] J. Król, K. Bielas, and T. Asselmeyer-Maluga. "Random World and Quantum Mechanics". In: *Foundations of Science* 28.2 (2023), pp. 575–625. DOI: 10.1007/s10699-022-09852-2

Some other co-authored, occasionally related publications include:

1. [28] K. Bielas, P. Klimasara, and J. Król. "The Structure of the Real Line in Quantum Mechanics and Cosmology". In: *Acta Physica Polonica B* 46 (Nov. 2015), p. 2375. DOI: 10.5506/APhysPolB.46.2375
2. [103] P. Klimasara et al. "The Latent Meaning of Forcing in Quantum Mechanics". In: *Acta Phys. Polon. B* 47 (2016), p. 1685. DOI: 10.5506/APhysPolB.47.1685
3. [112] J. Król et al. "Dimension 4: Quantum Origins of Spacetime Smoothness". In: *Acta Physica Polonica B* 48 (Jan. 2017), p. 2375. DOI: 10.5506/APhysPolB.48.2375

Chapter 2

Mathematical toolbox

This chapter serves as a longer, though not comprehensive, mathematical introduction to the thesis topics. It is intended to provide a background needed for Chapters 3, 4, 5, 6. Please note that the discussion related to some definitions, lemmas and other more detailed issues has been moved to Appendices A, B to ensure it does not disrupt the main line of reasoning.

2.1 Mathematics of quantum mechanics

2.1.1 The algebraic approach

As previously noted in Chapter 1, the conventional and first-choice approach to quantum mechanics involves Hilbert spaces of states and self-adjoint operators that represent physical, measurable quantities. In the folowing sections we introduce basic concepts and establish their relations with an algebraic approach, that will be more relevant in the rest of the thesis. To begin, it may be advantageous to briefly recall the most commonly used (and sometimes oversimplified) formulation of quantum mechanics axioms, following [33] (with some supplementary comments):

1. Given a quantum system, its attainable (pure) states are represented by unit vectors (rays, i.e. elements of the projective space) in a complex Hilbert space $\mathcal{H}$. For mixed states, one generalizes rays to the so-called density operators, i.e. self-adjoint, trace-class positive operators $\rho : \mathcal{H} \to \mathcal{H}$ such that $\mathrm{Tr}(\rho) = 1$.

   Remark 3. All physically relevant Hilbert spaces are usually assumed to be separable.

   Remark 4. The set of all states forms a convex set $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, where the set $\mathcal{S}_p \subseteq \mathcal{S}$ of extreme points coincides with one-dimensional projections on $\mathcal{H}$, which agrees with the preceding characterization of pure states by rays in $\mathcal{H}$. On the other hand, the set $\mathcal{S}_m \subseteq \mathcal{S}$ of non-pure, mixed states comprises all states ρ such that

   $$\rho(A) = t\rho_1(A) + (1-t)\rho_2(A)$$

   for all $t \in (0,1)$, $A \in \mathcal{B}(\mathcal{H})$ and $\rho_1 \neq \rho_2$.

2. Physical quantities (observables) are represented by self-adjoint operators $A : \mathcal{H} \to \mathcal{H}$.

   Remark 5. Some observables demand infinite-dimensional Hilbert spaces, e.g. the position Q and momentum P operators in (1.2) have to be unbounded and by the trace argument one easily realizes it is impossible to satisfy (1.2) in finite dimensions. Also, by Hellinger–Toeplitz theorem [152] unbounded operators cannot be defined on the whole $\mathcal{H}$, thus one usually restricts the domain to some dense subspace $D(\mathcal{H})$ of $\mathcal{H}$.

3. A pair (A, ρ) determines the probability distribution for measurement of the observable A in the state ρ, given by the *Born rule*:

$$\omega_\rho(A) = \mathrm{Tr}(\rho A). \tag{2.1}$$

Remark 6. In particular, if $\omega_\rho(A) = (\psi, A\psi)$ for some $\psi \in \mathcal{H}$ (i.e. ρ is pure, written also as $\rho = |\psi\rangle\langle\psi|$) and $\{\phi_i\} \subseteq \mathcal{H}$ is an eigenbasis of A with eigenvalues $\{\lambda_i\}$, one may expand as

$$\psi = \sum_i c_i \phi_i,$$

and the probability distribution of results λ while measuring A in a state ψ is of the form

$$\omega_\psi(A) = \sum_i \lambda_i \big|(\psi, \phi_i)\big|^2 = \sum_i \lambda_i |c_i|^2. \tag{2.2}$$

Then, measuring A in a state ψ will result in an eigenvalue λ_i with the probability $|c_i|^2$.

4. The evolution of a (autonomous) system is described by a unitary operator (actually, a one-parameter group of operators) $U(t) : \mathcal{H} \to \mathcal{H}$ defined by

$$U(t) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} tH},$$

where $H : \mathcal{H} \to \mathcal{H}$ denotes the Hamiltonian. This stands in striking contrast to system's evolution in case of a measurement procedure, which involves the so-called wave function collapse.

Remark 7. For an isolated state $\rho(0)$ at time $t = 0$, its state at some further time t is

$$\rho(t) = U(t)\rho(0)U(t)^{-1}.$$

Remark 8. In general, by Stone's theorem [152] there is a $1-1$ correspondence between self-adjoint operators and one-parameter groups of unitaries, given by $A \mapsto \mathrm{e}^{\mathrm{i}tA}$.

5. Given two systems with corresponding Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the composite system is described by tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Remark 9. One of the peculiar properties of a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is that not all the elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ are of the form $\psi_1 \otimes \psi_2$, i.e. there exist $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ that cannot be decomposed as $\psi = \psi_1 \otimes \psi_2$ with $\psi_1 \in \mathcal{H}_1$, $\psi_2 \in \mathcal{H}_2$; we call such states as non-decomposable, or entangled. Note that there is no counterpart of these in the case of classical physics, where composite systems are defined through elements of cartesian product, that are always decomposable.

Remark 10. The measurement process is canonically described as a two-stage operation [178][116]: given the quantum system S and measuring apparatus A, along with their initial states $|\psi_S\rangle \in \mathcal{H}_S$ and $|\psi_A\rangle \in \mathcal{H}_A$, their product state $|\psi_{SA}\rangle$ lives in the tensor product space $\mathcal{H}_S \otimes \mathcal{H}_A$ assigned to the system composed of S and A. Then, the process traditionally follows the rules:

$$|\psi_{SA}\rangle = |\psi_S\rangle |a_0\rangle = \sum c_i |\psi_i\rangle |a_0\rangle \xrightarrow{\text{stage 1}} \sum c_i |\psi_i\rangle |a_i\rangle \tag{2.3}$$

During stage 1, the system S and apparatus A interact, which turns $|\psi_{SA}\rangle$ into a mixed state. By (2.3) we see that the eigenstates $|\psi_i\rangle, |a_i\rangle$ a an observable on S and a pointer observable, respectively, become one-to-one correlated with each other. As we know, the actual measurement picks one of the possibilities out of the sum (2.3), what constitutes stage 2:

$$\sum c_i |\phi_i\rangle |a_i\rangle \xrightarrow{\text{stage 2}} |\phi_n\rangle |a_n\rangle \tag{2.4}$$

Observe that this probability picture is postulated in principle, and there is no known physical justification giving the reason for such a phenomenon. This point will be raised in the Chapter 6.

6. Symmetries of a system are unitary operators $\{V_i\}_{i\in I}$ commuting with H, i.e.

$$[V_i, H] = 0.$$

The following example both illustrates some of the above and makes a good starting point to discuss further implications such as Bell's inequalities or randomness (see Chapter 6).

Example 1. Consider a qubit, i.e. a quantum system with a corresponding Hilbert space $\mathbb{C}^2$, which may be realized practically e.g. by an electron's spin (as the name suggests, a qubit is a quantum version of the classical bit, i.e. the system with two possible states). Consider two bases of $\mathbb{C}^2$: the standard one: $\{|0\rangle, |1\rangle\}$ and another: $\{|+\rangle, |-\rangle\}$, defined by

$$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}. \tag{2.5}$$

Note that above bases consist of eigenvectors of respective Pauli matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

that correspond to spin's value along z and x, respectively. It is then easy to show that, preparing the spin in the state $|+\rangle$ (i.e. spin up in x direction) and measuring it in z direction, one obtains either $|0\rangle$ or $|1\rangle$, both with probability $\frac{1}{2}$, due to Born rule (2.2).

Consider now the system composed of two spins (subsystems) A and B, described by Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. The state

$$|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}} \equiv \frac{|1_A 0_B\rangle - |0_A 1_B\rangle}{\sqrt{2}}$$

is the example of an entangled state. It is also a state that clearly shows the implications of the existence of entangled states. Namely, consider a scenario where two parties (Alice and Bob) perform the space-like separated measurements of spin in z direction; in particular, if Alice measures a value $+1$ (-1), then the system collapses into $|0_A 1_B\rangle$ ($|1_A 0_B\rangle$). This causes Bob's subsystem to collapse immediately after Alice's measurement, regardless of the separation of their laboratories. Although the instantaneous result seems to contradict special relativity, particularly causality (cf. the seminal work [60]), a more careful analysis shows quantum mechanics avoids precisely such problematic issues, and neither theoretical nor experimental evidence has been presented to challenge special relativity thus far. Even if Bob attempts to extract information from the way Alice makes the measurements, his results will remain purely probabilistic, as quantum mechanics predicts. We elaborate on this issue further in this section.

We have encountered the conventional framework of quantum mechanics from the perspective of Hilbert spaces, and the way it applies to a very simple physical case. Let us introduce an equivalent yet more abstract approach, suited to model-theoretic and categorical picture: the operator algebra approach. The theory of operator algebras, introduced by von Neumann with the aim of providing a solid mathematical foundation for quantum mechanics [142], quickly started to live its own life and gave rise to many new notions including von Neumann algebras [139] and more broadly C^*-algebras [68], finally planting seeds for algebraic quantum field theory [76] and noncommutative geometry [49]. As we have observed, in the standard formulation of quantum mechanics the prevailing logic dictates that observables act on states. The foundational principle of the algebraic approach to quantum mechanics is quite the opposite: we begin with the algebra of operators-observables as the starting point and then states are to be defined upon them. This "duality" can be noticed e.g. in the trace prescription (2.1) and will be made precise in the following.

We start with an abstract, unital C^* algebra A, i.e. a Banach algebra equipped with an involution $* : A \to A$, i.e. an endomorphism for which

$$\|a^*a\| = \|a\|^2, \, a \in A.$$

Example 2. We have some straightforward realizations of an abstract C^*-algebra (both classical and quantum one):

- the algebra $C_0(X)$ of continuous functions $f : X \to \mathbb{C}$ that vanish at infinity for a locally compact Hausdorff space X; an involution is obviously given by pointwise complex conjugation and
$$\|f\|_\infty = \sup_{x \in X}\{|f(x)|\}$$
- the algebra $\mathcal{B}(\mathcal{H})$, where an involution is given by the adjoint and
$$\|a\| = \sup_{\psi \in \mathcal{H}}\{\|a\psi\|\}$$

In fact, above examples essentially exhaust concrete realizations of an abstract C^*-algebra, as it holds [63]

Theorem 1. (Gelfand–Naimark) Every commutative C^*-algebra A is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X and such X is unique up to homeomorphism. More generally, every (commutative or noncommutative) C^*-algebra is isomorphic to a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

We start with a physical assumption that observables are to be represented by self-adjoint elements of a C^*-algebra A, and it makes sense now to define states as functionals $\omega : A \to \mathbb{C}$ that are positive and normalized, i.e. $\omega(a^*a) \geq 0$ and $\omega(1) = 1$. In particular, it holds that $\omega(a) \in \mathbb{R}$ whenever $a \in A$ is self-adjoint. It is also easy to see that (2.1) gives a map $\omega_\rho : A \to \mathbb{C}$ that is linear, positive and normed, hence is a state in the latter sense. Recall that $p \in A$ is called a projection if $p^2 = p^* = p$; if $A = \mathcal{B}(\mathcal{H})$, we denote the set of projections on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and there is a $1-1$ correspondence between $\mathcal{L}(\mathcal{H})$ and closed linear subspaces of $\mathcal{H}$ given by $\mathcal{L}(\mathcal{H}) \ni p \mapsto \mathrm{ran}(p) \subseteq \mathcal{H}$. It follows immediately that the only eigenvalues of a projection are 0 and 1. By (2.2), projections seem to be good candidates for yes-no questions, such as "Is the photon polarized vertically?" or "Does the electron have spin up in the z-direction?" etc.

Now, the connection of these approaches can be shown through Gleason's theorem [116]:

Theorem 2. Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) > 2$. Then every probability measure μ on $\mathcal{L}(\mathcal{H})$ is induced by a normal state on $\mathcal{B}(\mathcal{H})$ by

$$\mu(P) = \mathrm{Tr}(\rho P), \tag{2.6}$$

where ρ is a density operator, uniquely determined by μ. Conversely, every density operator $\rho \in \mathcal{B}(\mathcal{H})$ defines a probability measure $\mu : \mathcal{L}(\mathcal{H}) \to [0, 1]$ through (2.6).

Even if elements of $\mathcal{B}(\mathcal{H})$ hardly seem to represent anything real, there is a context that gives a precise interpretation of observables represented by self-adjoint operators: the spectral theorem. First we provide the usual formulation of the theorem [170]:

Theorem 3. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint; then there exists a measure space (X, μ), a bounded measurable function $f : X \to \mathbb{R}$ and an isomorphism $U : \mathcal{H} \to L^2(X, \mu)$ such that

$$A = U^{-1} M_f U,$$

where $M_f : L^2(X, \mu) \to L^2(X, \mu)$ is defined by

$$M_f(g) = f \cdot g.$$

Thus every self-adjoint A operating on $\mathcal{H}$ can be represented by some $L^2(X, \mu)$ together with a measurable, real-valued function; in other words, the action of A on some $\psi \in \mathcal{H}$ is equivalent to transforming ψ to a square-integrable function on X first, then multiplying by a unique $f \in L^2(X, \mu)$ and going back to another $\phi \in \mathcal{H}$ finally.

Now that we are equipped with some algebraic perspective on quantum mechanics, let us turn our attention to the structure of projections in $\mathcal{B}(\mathcal{H})$ due to their role in constructing self-adjoint elements of $\mathcal{B}(\mathcal{H})$ and the logic of quantum mechanics. As discussed earlier, projections serve as "yes-no" questions. Recall that in classical physics such questions are in one-to-one correspondence with measurable subsets of the phase space, and these build up a Boolean propositional logic. By analogy, one might wonder whether it is possible to formulate a similar kind of logical structure connected to $\mathcal{L}(\mathcal{H})$. The answer is affirmative: once again we will use the correspondence $p \mapsto \mathrm{ran}(p)$ between projections and closed subspaces of $\mathcal{H}$. Then, the set $\mathcal{L}(\mathcal{H})$ equipped with

1. a join
$$\bigvee p_i = p_M \text{ where } M \text{ is the closed linear span of } \bigcup \mathrm{ran}(p_i), \tag{2.7}$$

2. a meet
$$\bigwedge p_i = p_M \text{ where } M = \bigcap \mathrm{ran}(p_i), \tag{2.8}$$

3. a negation $\neg p = 1 - p$,

4. a null operator as 0 and identity operator as 1.

gives the structure $(\mathcal{L}(\mathcal{H}), \vee, \wedge, \neg, 0, 1)$, which is known to be an orthomodular lattice (see A. Interpreting symbols $\vee, \wedge, \neg, 0, 1$ as connectives "or", "and", "not", true and false, respectively, we build propositions on the system in the language of the so-called quantum logic, proposed first in [31]. Observe that it is not immediately evident where all the unique quantum properties manifest within the framework of $\mathcal{L}(\mathcal{H})$. Indeed, one might argue whether the noncommutativity of observables is transferred to the logical structure of $\mathcal{L}(\mathcal{H})$. The answer is affirmative, as one finds [142]

Lemma 1. If $\dim(\mathcal{H}) \geqslant 2$, the lattice $\mathcal{L}(\mathcal{H})$ is not distributive and hence fails to be a Boolean algebra.

Thus, as long as we consider physically plausible state spaces, i.e. $\dim(\mathcal{H}) \neq 1$, it is generically inevitable for the global structure of $\mathcal{L}(\mathcal{H})$ to be nondistributive, failing to be Boolean in particular. On the other hand, one proves easily[63]

Lemma 2. Let A be a unital abelian C^*-algebra. Then $\mathcal{L}(A)$ is a Boolean algebra.

We have already mentioned that systems represented by infinite-dimensional Hilbert spaces will be of major interest, mainly due to operators of continuum spectrum, such as position or momentum, and the respective uncertainty relation. Therefore by Lemma 1, $\mathcal{L}(\mathcal{H})$ will be generically nondistributive. It is instructive to give a simple example illustrating the nondistributivity of $\mathcal{L}(\mathcal{H})$ for a specific quantum system[46].

Example 3. Let $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}x)$ and $p_1, p_2, p_3 \in \mathcal{B}(\mathcal{H})$ be mutually non-orthogonal, one-dimensional projections contained in a single, closed two-dimensional subspace (e.g. they can project on three mutually non-orthogonal directions lying on a plane in $\mathbb{R}^3$). Define the join $\vee$ and meet $\wedge$ as in (2.7), (2.8); now it is easy to see that

$$p_1 \wedge (p_2 \vee p_3) = P_1 \neq 0 = (p_1 \wedge p_2) \vee (p_1 \wedge p_3),$$

which obviously shows that $\mathcal{L}(\mathcal{H})$ is not distributive.

Remark 11. Note that $\mathcal{L}(\mathcal{H})$, while non-distributive, still obeys the law of excluded middle, since

$$p \vee \neg p = 1$$

by (2.7). Thus, one may conclude that such a "logic" is simultaneously peculiar (non-distributivity) and conventional (excluded middle satisfied) at the same time (these aspects will be "turned over" in quantum logic formulated within some particular topoi, see Section 2.2.3).

Let us now introduce two results that illustrate the significant departure of quantum mechanics from classical physics when it comes to our understanding of "reality": the Kochen–Specker theorem and Bell inequalities. First, certain experiments, such as the Stern–Gerlach experiment, indicate that the preexistence of well-defined values of physical quantities can be troublesome. In fact, this idea can be precisely defined and it serves as yet another boundary between classical and quantum physics. The property of quantum mechanics characterized by the relational nature of reality, i.e. that measurements do not actually "reveal" the actual values of quantities that exist independently of the measuring "context", is known as *contextuality* and is closely related to the hidden variables (HV) theories concept. We make it more precise with the help of a *valuation* function $v : X \to \mathbb{R}$ that, for a given state ψ, sends a physical quantity f to its value λ in this state. Classically we have therefore

$$C^\infty(X) \ni f \mapsto v_\psi(f) = f(\psi) \in \mathbb{R},$$

where we identified a physical quantity with the associated function f (see Chapter 1). However, in quantum mechanics this pairing becomes

$$\mathcal{B}(\mathcal{H}) \ni a \mapsto v(a) \in \sigma(a) \subseteq \mathbb{R},$$

where $\sigma(a)$ is the spectrum of a. It can be shown that there is a particular condition, called functional composition condition (FUNC), that constrains a valuation v to be *mutually exclusive* and *collectively exhaustive*, meaning that given a resolution of unity

$$1 = \sum p_i, \quad p_i^2 = p_i,$$

it holds $v(p_i) = 1$ for some $i \in I$ and $v(p_j) = 0$ for all $j \neq i$. The FUNC condition reads [65]

$$\forall h : \mathbb{R} \to \mathbb{R} \left(v(h(a)) = h(v(a)) \right), \quad a \in \mathcal{B}\left(\mathcal{H}\right). \tag{2.9}$$

This way one states [65]

Theorem 4. If $\dim(\mathcal{H}) > 2$, then there does not exist any valuation function $V : \mathcal{B}\left(\mathcal{H}\right) \to \mathbb{R}$ such that (2.9) is satisfied for all $a \in \mathcal{B}\left(\mathcal{H}\right)$.

Informally, Kochen–Specker theorem states that as long as the state space is more than two-dimensional, there is no function that would globally assign real values to physical quantities in a noncontextual way. In other words, physical quantities in their totality have no "objective" value prior to measurement. Further we will see that Kochen–Specker theorem takes particularly simple and intuitive form from the topos-theoretic perspective. One should note that the original formulation of Kochen–Specker theorem [104] involves a variant of a Boolean algebra, that takes into account the noncompatibility of observables. Define a partial Boolean algebra A to be a set with constants $0, 1$ and a reflexive, symmetric binary compatibility relation $\odot \subseteq A \times A$, a total unary operation $\neg$ and partial binary operations $\wedge, \vee$ defined on $\odot$, and every set of pairwise compatible elements is contained in at least one Boolean algebra $B \subseteq A$. In Chapter 5 we will see how the class of partial Boolean algebras can be made into a category **pBA**.

Example 4. One of the most important examples of a partial Boolean algebra is $\mathcal{L}\left(\mathcal{H}\right)$. Then, the compatibility of $p, q \in \mathcal{L}\left(\mathcal{H}\right)$ corresponds precisely to their commutativity.

Then we have [104]

Theorem 5. There is no embedding of $\mathcal{L}\left(\mathcal{H}\right)$ into a Boolean algebra if $\dim(\mathcal{H}) \geq 3$.

Theorems 4, 5 bring to the scene the *problem of hidden variables*. Historically, from the very beginning of quantum theory there has been a live debate on the existence of a non-quantum, deterministic reality underlying a supposedly incomplete, probabilistic microworld, initially centering around the work of Einstein, Podolsky and Rosen [60]. Such a reality would consist of the set Λ of unobservable degrees of freedom such that every $\lambda \in \Lambda$ determines a dispersion-free measure on $\mathcal{L}\left(\mathcal{H}\right)$. Importantly, such parameters $\lambda \in \Lambda$, called non-contextual, were refuted by Theorem 2. However, it was shown that it was still possible to construct dispersion-free measures as long as they were defined only on maximal Boolean subalgebras of $\mathcal{L}\left(\mathcal{H}\right)$ [74]; these variables are called contextual

Remark 12. Up to this point, we have used the term "random" on multiple occasions in an informal manner, relying on its intuitive understanding. For instance, in the context of the Stern–Gerlach experiment we discussed within the Copenhagen interpretation, it seems to provide a satisfactory explanation of how we characterize a property (specifically, spin) as random. The notion of randomness in quantum mechanics is profoundly captivating and surprisingly intertwined with the upcoming section dedicated to model theory, which we will delve into separately in Chapter 6.

2.2 Model theory and categories

Now we introduce formal methods, that enable us to discuss the shift from "global" mathematics into more "local" one. This section provides also a more solid underpinning for the related discussion given in Chapter 1 on the approaches to real numbers. In particular, we mentioned that it would be highly desirable to transition from one "type" of real numbers to another, giving the possibility to capture various problems connected to crossing the micro-macro boundary. In the following section, we will provide a general introduction to model- and category-theoretic tools employed throughout the thesis, along with a brief account on the history and respective literature. We try to be as rigorous as possible; however, in order to keep the discussion instructive, some technical details are put aside to Appendix B. Also, to get a more formal background on both model theory and category theory in general, we refer the reader again to Appendix B.

2.2.1 Forcing and Boolean-valued models

As the forcing method is one of the thesis' central topics, we give a more detailed account on that subject. To truly appreciate the significance of independence results, we need to go back perhaps to Cantor, who was the first to demonstrate that any set X is essentially smaller than its powerset $\mathcal{P}(X)$, i.e. there is no surjection X onto $\mathcal{P}(X)$, what is now widely recognized as Cantor's theorem [43]. Given that $\mathbb{R}$ can be identified with the powerset $\mathcal{P}(\mathbb{N})$ (for any $r \in \mathbb{R}$, think about digits in the decimal expansion of r as the subset of $\mathbb{N}$), since the time of Cantor it has been known that

$$2^{\aleph_0} > \aleph_0,$$

where $\aleph_0 = |\mathbb{N}|$ and $2^{\aleph_0} = |\mathbb{R}|$. However, whether the continuum is the next cardinal number, i.e.

$$2^{\aleph_0} = \aleph_1, \tag{2.10}$$

or perhaps

$$2^{\aleph_0} = \aleph_i \text{ for some } i > 1 \tag{2.11}$$

and thus such $\aleph_i$ lived "somewhere in the middle", was far from obvious. In fact, the statement (2.10), subsequently known as the Continuum Hypothesis (CH), was considered so profound that Hilbert placed it on the very top of his list of most-important, unsolved mathematical problems back in 1900 [83]. (It is worth noting that another problem on the list was Hilbert's aspiration to provide consistent axiomatization of arithmetic.) Meanwhile, substantial progress has been made when K. Gödel demonstrated that every sufficiently powerful formal system is either incomplete or inconsistent [70], and Gödel's result proved that Hilbert's dream of settling the "perfect" foundations for mathematics had to be abandoned. Essentially, as long as one considers theories like ZFC or Peano arithmetic to be consistent (as we often implicitly do), undecidable statements (sentences one cannot prove either true or false) will persist.

When discussing truth, one needs to be careful and distinguish between the words *true* and *provable*, as they operate on two distinct levels: semantic (model) and syntactic (theory) (see Appendix B). For instance, soundness tells us that given a first-order theory (like ZFC), every provable sentence is true in all models. Conversely, given a first-order theory T, there exist sentences true in some models of T and false in the others. These sentences are termed *undecidable*, since one cannot decide their truth from T alone, and they represent peculiarities of particular models rather than the theory itself. A somewhat iconic example might be Euclid's fifth axiom, called *the postulate of parallel lines*, that is independent of the other four axioms of Euclidean geometry: an independence of the fifth axiom becomes clear once we construct two models — the Euclidean plane which satisfies all five axioms and the hyperbolic plane, satisfying all but exactly the fifth axiom.

Coming back to CH, it seems that the hint of independence is already present in the Löwenheim–Skolem theorem (see Theorem 40). This theorem asserts that the existence of an infinite ZFC model implies the existence of a countable one as well. Obviously, it may initially seem paradoxical, since inside each countable model M one finds a set of real numbers $\mathbb{R}_M$ expected to be uncountable. This phenomenon, known as Skolem's paradox, becomes clear as long as one keeps in mind that a set X is claimed to be uncountable in the model M if there is no surjection $\mathbb{N} \to X$ in M. Consequently, uncountability is not an absolute property and may vary, simply by conceiving it from within or outside a specific model. Now, despite Gödel had shown $\mathrm{Con}(\mathrm{ZFC} + \mathrm{CH})$ via the constructible ZFC model L satisfying CH [70], i.e.

$$L \models \mathrm{ZFC} + \mathrm{CH} \tag{2.12}$$

it was not immediately evident whether it was possible to "design" a ZFC model satisfying $\neg$CH, see (2.11). Thus, it gained considerable attention of the mathematics community when P. Cohen presented *forcing*, a powerful technique of proving independence results by constructing model extensions with desired properties [47]. In the case of CH, P. Cohen used forcing to develop a model extension $M[G]$ of a ZFC model M such that

$$M[G] \models \mathrm{ZFC} + \neg\mathrm{CH}. \tag{2.13}$$

Roughly speaking, the technique involved adding $\aleph_2$-many "missing" subsets of natural numbers, in order to obtain

$$2^{\aleph_0} \geq \aleph_2 > \aleph_1. \tag{2.14}$$

Note that (2.14) contradicts (2.12), thus we obtain (2.13) and $\mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH})$. Naturally, it is quite nontrivial to ensure that, starting with a ZFC model M, the extension $M[G]$ still satisfies ZFC. Since Gödel already observed $\mathrm{Con}(\mathrm{ZFC} + \mathrm{CH})$, we conclude that CH has to be independent of ZFC.

Let us briefly outline how the above extension actually works (cf. Appendix B for details). As previously mentioned, we begin with a ZFC model M (typically chosen as countable and transitive for practical reasons). What Cohen discovered was a systematic way to construct new, extended models $M[G]$ derived from M, by identifying a specific set $G \subset M$, $G \notin M$. This G is designed to provide $M[G]$ to be a unique, minimal ZFC model and containing all the elements of M including $G \in M[G]$. In order to have a control of what sentences are true in $M[G]$, one picks a specific separative partial order $P \in M$ (called a forcing notion); each $p \in P$ (called a forcing condition) represents a piece of knowledge about $M[G]$, and $p \leq q$ means that p gives more information than q ($q \subseteq p$). In order to control the extension $M[G]$, we demand G to be a *generic* ultrafilter, i.e. $G \cap D \neq \varnothing$ for every $D \in M$, $D \subseteq P$ that is dense in P. Since G is an ultrafilter, one interprets its elements as those specifying truth, thus such G does not introduce any inconsistencies; as G is generic, it gives as much information about $M[G]$ as possible. It is important to stress that the properties of $M[G]$ are "accessible" from within M with the help of the so-called *forcing language*, expressing sentences containing P-names, i.e. the sets of "potential" elements composed of "potential" elements etc. (Although M decides the truth value of sentences in $M[G]$ upon G, and does it decisively only for sentences that are either true or false in M.) Once the ultrafilter G is chosen, G gives an interpretation for each P-name, which constitutes $M[G]$. As expected, $M[G]$ is a countable transitive ZFC model with $M \subseteq M[G]$ and $G \in M[G]$. Then, we say *p forces ϕ* (written $p \Vdash \phi$) if $M \models \phi$ for every generic ultrafilter G with $p \in G$. Informally, if $p \Vdash \phi$, then $p \in G$ is a sufficient "piece of information " about ϕ in order to $M[G] \models \phi$.

Just a few years after Cohen's method was introduced, Scott and Solovay observed that the entire procedure can be replicated by shifting truth-values from ordinary two-valued $\{\text{True}, \text{False}\} \equiv \{0, 1\}$ to a more general, complete Boolean algebra $B = P \cup \{0\}$ (cf. Appendix B). This change replaces the universe V of standard sets with the universe V^B of their B-valued counterparts, where every formula ϕ in the language of set theory is assigned a truth value $[\![\phi]\!] \in B$. This reflects the "partial knowledge" of what should be true in the extended model, similarly to P-names described above. Still, every theorem ϕ of ZFC remains true in V^B (see Theorem 41). Crucially, to obtain a two-valued ZFC model, a suitable ultrafilter $U \subseteq B$ is identified to represent "truth", i.e. we take the quotient V^B/U, identifying the elements of x, y in V^B whenever $[\![x = y]\!] \in U$. This way one shows that V^B/U is a standard ZFC model, usually collapsed by Mostowski's theorem to a transitive ZFC model $V[U]$. (Note that in order to ensure that generic ultrafilters exist in general, we often assume the model to be countable.) Notably, forcing aquires a natural interpretation in the language of Boolean-valued models:

$$p \Vdash \phi \text{ iff } p \leq [\![\phi]\!]^B$$

The introduction provided above clearly illustrates that throughout the 20th century, mathematics developed numerous tools to make reasoning about physics "more local": one is no longer confined to a single universe V, but can instead employ a multitude of "small" universes. Thus several important questions arise: what rules dictated the choice of models, how these models are interconnected and whether this has any relevance to physics? This is the point where we return to forcing: it not only offers a means to establish independence results, but also provides a method for constructing ZFC models in a controlled and systematic manner.

2.2.2 Model theory and physics

The first notable attempt to formally incorporate model theory in physics was perhaps the work of P. Benioff [23, 24], soon after P. Cohen proved that neither CH nor AC are provable from ZF. Benioff's key insight was to challenge the conventional practice in physics, which typically uses mathematical objects as they would live in a single set-theoretic universe (often denoted V). Let us outline the approach to understand the quantum world through countable ZFC models as advocated in Benioff's work, elaborating on the issue of formulating physics, i.e. the mathematical structures representing physical objects — spaces, observables, states etc. — in formal models of set-theory, rather than to treat all the physics "uniformly", as living in the single, default universe of sets V. Under specific assumptions, this line of reasoning led Benioff to two main observations:

B.1 The minimal, standard ZFC model M_0 cannot be a carrier for the mathematics of quantum mechanics (assuming the outcomes of infinitely repeated measurement of a quantum-mechanical system leads to a sequence that is random in a certain sense).

B.2 No Cohen extension of M_0 can serve as such a carrier (assuming additionally the infinite sequences of outcomes to be statistically independent in a certain sense).

In this context, we define a ZFC model M as a "carrier of mathematics of quantum mechanics if all relevant structures such as Hilbert spaces, algebras of observables etc. can be consistently formulated inside M *and* the outcomes of infinitely repeated measurements are also accessible in M. We will see that the second criterion heavily depends on the actual definitions of randomness and independence; the results of [23, 24] summarized above hold true only if adopted definitions are sufficiently strong. In the subsequent discussion we provide a more detailed discussion of the above, setting the stage for an exploration of the issue of randomness in quantum mechanics, which is a main topic of Chapter 6.

Let M be a countable transitive ZFC model and let $\mathcal{H}_M, \mathcal{B}(\mathcal{H}_M)$ be a Hilbert space and a set of bounded operators on $\mathcal{H}_M$ inside M, respectively. More precisely, if $\phi(x)$ represents a formula "x is a Hilbert space", then we have $M \models \phi(\mathcal{H}_M)$ and similarly for $\mathcal{B}(\mathcal{H}_M)$. Note that $\phi(x)$ is not M-absolute, thus $\mathcal{H}_M$ is not necessarily a Hilbert space outside M. In fact, one proves [23]

Theorem 6. Let M be a standard transitive ZFC model. Then there exists a Hilbert space $\mathcal{H}$ and a set of bounded operators $\mathcal{B}(\mathcal{H})$ (outside M) such that there exist isometric monomorphisms $U_M : \mathcal{H}_M \to \mathcal{H}$ and $V_M : \mathcal{B}(\mathcal{H}_M) \to \mathcal{B}(\mathcal{H})$. In particular it holds that V_M maps projections, unitary, self-adjoint and density operators in M to respective operators outside M, e.g.

$$M \models (p \text{ is a projection } \implies V_M(p) \text{ is a projection})$$

Obviously both $U_M, V_M \notin M$, although all the reasoning about $\mathcal{H}$ outside M holds for $\mathcal{H}_M$ inside M.

Let $\mathbf{S}, \mathbf{Q}$ denote the collections of state preparation and question measuring procedures, respectively: these correspond essentially to density operators and projections in $\mathcal{B}(\mathcal{H}_M)$. In other words, one introduces the corresponding maps $\psi_M : \mathbf{S} \to \mathcal{B}(\mathcal{H}_M)$ and $\phi_M : \mathbf{Q} \to \mathcal{B}(\mathcal{H}_M)$. Therefore, $\psi_M(a)$ and $\phi_M(b)$ represent a density operator (state preparation) and projection (a yes-no question) in $\mathcal{B}(\mathcal{H}_M)$, respectively. Consequently, the process of (infinite) measurement repetition can be defined by a triple (t, a, b) with $t : \omega \to \mathbb{R}_M$ and this map is interpreted as: "prepare the state according to $a \in \mathbf{S}$, carry out the question $b \in \mathbf{Q}$, observe an outcome, discard the system and repeat the above at $t(0), t(1), \ldots$. Now, with the help of monomorphisms U_M, V_M given in Theorem 6, one shows that

$$\forall a \in \mathbf{S}_M,\, b \in \mathbf{Q}_M \left(\mathrm{Tr}_M \left(\psi_M(a)\phi_M(b) \right) = \mathrm{Tr} \left(\psi(a)\phi(b) \right) \right). \tag{2.15}$$

Note that $\mathbf{S}_M, \mathbf{Q}_M$ are $\mathbf{S}, \mathbf{Q}$ inside M, and due to (2.1), the equality (2.15) states that the observers inside and outside M will obtain exactly the same measurement outcomes.

Remark 13. It is important to note that one can replace the universe of sets V by a standard transitive ZFC model N such that $M \subsetneq N$ with all the conclusions holding true; in particular, given $\mathcal{H}_M$ and $\mathcal{B}(\mathcal{H}_M)$ there exist $\mathcal{H}_N$ and $\mathcal{B}(\mathcal{H}_N)$ such that there exist isometric isomorphisms

$$U_{MN} : \mathcal{H}_M \to \mathcal{H}_N, \quad V_{MN} : \mathcal{B}(\mathcal{H}_M) \to \mathcal{B}(\mathcal{H}_N)$$

and

$$\forall a \in \mathbf{S}_M,\, b \in \mathbf{Q}_M \left(\mathrm{Tr}_M \left(\psi_M(a)\phi_M(b) \right) = \mathrm{Tr}_N \left(\psi_N(a)\phi_N(b) \right) \right). \tag{2.16}$$

The identity (2.16) suggests that ZFC models are essentially equivalent as universes for mathematics of quantum mechanics in that the observable outcomes are the same as long as they are interpreted inside models M, N such that $M \subseteq N$. However, this is not true in general, what brings us to **B.1**.

In order to express **B.1** precisely, recall that the minimal model M_0 is constructed as follows. Let δ be the smallest ordinal for which there exists a standard transitive ZFC model M such that $\delta \notin M$. Then one shows that

$$M_0 = \bigcup_{\alpha < \delta} L_\alpha$$

is the unique standard ZFC model that is minimal in the sense that for any standard transitive ZFC model N we have $M_0 \subseteq N$ (recall that L_α is a stage of constructible universe, see Remark 50).

Definition 1. We say that a probability measure $\mu : \mathrm{Bor}(2^\omega) \to \mathbb{R}_{\geq 0}$ is *correct* for $\sigma \in 2^\omega$ if for all $B \in \mathrm{Bor}(2^\omega)$ definable from μ, if $\mu(B) = 1$ then $\sigma \in B$. A sequence $\sigma \in 2^\omega$ is called *random* if there exists a probability measure μ that is correct for σ.

Informally, the sequence is random if there exists a probability measure such that σ falls in every definable Borel subset of full measure, and such Borel subsets may be interpreted to carry as little information about σ as possible. Therefore, the definition seems to fit well an intuition of what characteristics a random sequence should enjoy, and we will follow the topic extensively in Chapter 6. Finally, one proves [23]

Theorem 7. Suppose that the outcomes of a generic quantum-mechanical experiment are random in the sense of Definition 4. Then M_0 is not a possible mathematical universe for quantum mechanics in the sense that for a triple (t, s, q) defined above, the sequence of outcomes $\psi_{tsq} \notin M_0$.

For a Theorem 7 to hold, it is necessary to assume that the probability measure μ is nonatomic, i.e. $\mu(\sigma) = 0$ for every $\sigma \in 2^\omega$ or, for those s, b for which

$$0 < \mathrm{Tr}_{M_0}\left(\psi_{M_0}(s)\phi_{M_0}(b)\right) < 1.$$

We see that above condition excludes sequences constantly equal to 0 or 1, and these are just classical (dispersion-free) outcomes as long as $\psi_{M_0}(s)$ is pure. Therefore, M_0 is a possible universe for classical mechanics, contrary to the generic quantum case.

Finally, one obtains

Theorem 8. Let randomness and statistical independence be defined as above and let $s \in \mathrm{Dom}(\psi_{M_0})$, $q \in \mathrm{Dom}(\phi_{M_0})$ and $t : \omega \to \mathbb{R}_{M_0}$, $t \in M_0$. If ψ_{tsq} is the sequence of outcomes with repeated s, q, then the extension $M_0[\psi_{tsq}]$ cannot be a carrier for mathematics of quantum mechanics.

A different and less technical route of applying model theory to physics was taken by Davis [50], where he used the work of Takeuti [170] to show that Boolean-valued models provide a way to translate quantum-mechanical, noncommutative observables to classical, real-valued quantities, such as position or momentum. As outlined in Section 2.2.1, the point of departure is a complete Boolean algebra that serves as a collection of truth-values assigned to formulae describing the extension. In the case of quantum-mechanical systems, the natural choice for such collections are Boolean subalgebras of $\mathcal{L}(\mathcal{H})$ and indeed this choice appears frequently throughout the thesis. Due to one of the versions of spectral theorem (see Theorem 18), for every family of commuting self-adjoint operators $\{A_i\}$ there is a maximal Boolean algebra B of projections containing all the spectral resolutions of $\{A_i\}$ and we say $\{A_i\}$ are *contained* in B. The following fundamental result of [170] claims that the elements of $\{A_i\}$ can be "reinterpreted" as real quantities. (Note that real numbers inside V^B denoted by $\mathbb{R}_B$ is not an absolute object.)

Lemma 3. For every complete Boolean algebra $B \subseteq \mathcal{L}(\mathcal{H})$ of projections, there is a bijective correspondence between real numbers $\mathbb{R}_B$ in V^B and self-adjoint operators contained in B.

This gives an insight how one would formalize quantization in a model-theoretic language. Indeed, as observed first by Dirac [51], quantization can be understood as replacing the real-valued quantities such as position, momentum etc. with their noncommutative counterparts, where fundamental relations such as Hamilton equations of motion still hold, provided the substitution $\{\cdot,\cdot\} \to i\hbar[\cdot,\cdot]$ of Poisson bracket with a commutator. Accordingly, due to Lemma 3, for every Boolean algebra of operators one always finds a universe (Boolean-valued model) with real numbers bijectively represented by self-adjoint operators that are in B, which can be seen as a way to quantize a theory. This is in agreement with a general philosophy of perceiving quantum effects always through "classical glasses", hence Boolean algebras containing necessarily only mutually commutative operators. In [50] a specific "relativity principle" is proposed, in analogy to special relativity, where inertial frames become Boolean algebras of projections. According to this principle, non-intuitive effects in quantum mechanics are the results of considering quantum systems from within *different* "classical reference frames", i.e. Boolean algebras that do not commute with each other. To illustrate such a case, let us consider an experiment involving a measurement of noncommuting (complementary) quantities, e.g. position Q and momentum P in double-slit interference. Recall the peculiar behaviour here is distinct patterns on screen, that depend on the slit configuration, and it is easy to see that having one slit covered corresponds to the position measurement, while having both slits open measure a momentum. Thus, such measurements are done with respect to two reference frames $B_Q \ni Q$ and $B_P \ni P$ and there is *no* Boolean reference frame B_{QP} such that $Q, P \in B_{QP}$. Observe that the analogy with e.g. time dilation as an effect of measurement with respect to distinct inertial frames is evident here. As the analogy goes further, the role of Lorentz transformations is played by unitary operators (e.g. Fourier transform mapping position to momentum representation) and finally the four-vector $x \in \mathbb{R}^4$ corresponds to the wavefunction $\psi \in \mathcal{H}$ (with metric invariant $\mathrm{d}s^2$ replaced by the norm $\|\psi\|$).

Remark 14. Quite separately from the approach taken here, a few works discussing the direct importance of CH independence in physics have appeared, cf. [148], [147]. A discussion on selected topics in the set-theoretic foundations for physics can be found e.g. in [169][17][41][16].

2.2.3 Topoi

In the preceding section, we outlined why it is reasonable to question mathematical foundations in physics, and we illustrated this claim with some applications. The fundamental tool used there was forcing (both in the language of posets and Boolean algebras). Here we introduce an alternative approach to challenge these foundational assumptions: a topos, which is a particular category that serves as a framework for a significant portion of contemporary mathematics (see Appendix B). Remarkably, there exists a connection between Boolean-valued models and topoi: given a complete Boolean algebra B, a Boolean-valued model V^B is equivalent to a topos $\mathrm{Sh}(B)$ of sheaves over B. Before we dwelve into that, let us briefly describe general motivations that have brought topoi into the field of physics.

To begin with, let us refer to Grothendieck himself [117]: *These 'probability clouds', replacing the reassuring material particles of before, remind me strangely of the elusive "open neighborhoods" that populate the topoi, like evanescent phantoms, to surround the imaginary 'points'.* This quote offers a reflection on the phenomena discussed in our introduction to quantum mechanics. Indeed, as we approach smaller distances (and higher energies) in the experiments, quantum effects enter the stage and the values of physical quantities usually cease to be sharp and well-defined. Two most known approaches to quantum physics through topos theory evolved from a quite similar point of departure; the earlier one, called a contravariant approach (developed mainly by Isham and Döring (see e.g. [88]) and another, called a covariant approach (originated from the work [117] by Landsman, Heunen and Spitters) (this naming convention is borrowed from a review [175]). The reasoning could be characterized as follows:

- quantum mechanics needs a reformulation, perhaps in a way that enables one to conceive it as a "realist" theory,

- a quantum phase space Σ needs to be devised in a more "geometric" manner, along with states and observables defined with the help of Σ
- quantum "logic" needs to be modified to become a legitimate logic as a result

We will provide a brief overview of the the covariant approach, also called a *Bohrification* [82]. This approach addresses the challenges outlined above by relying on three pillars: topos theory, Gelfand duality and an algebraic approach to Bohr's perspective on quantum mechanics. It is worth recalling that Bohr emphasized the need to perceive quantum-mechanical (noncommutative in their nature) phenomena ultimately through classical (i.e. commutative) notions, cf. [117]. Thus, the primary objective is to give a rigorous foundation for Bohr's doctrine of classical concepts, asserting that all investigations in quantum world have to be captured through classical physics eventually. In mathematical terms, this entails approaching the noncommutative algebra of quantum–mechanical observables through its commutative subalgebras. Here, the authors of [81] have chosen to employ C^*-algebras, introduced briefly in Section 2.1.1, as the most natural and useful language. While it had been already recognized that topoi provide a powerful yet not entirely classical formal environment for quantum mechanics [88], it was the application of Gelfand duality in topoi [137] that gave a new insight into spatial aspects of the algebra of observables. Before we delve into the specific construction, let us show intuitively how topos theory enters the stage through some observations concerning the logic of classical and quantum physics from category-theoretic perspective [80].

Remark 15. As we previously described, in classical physics the description of a given system is provided by a phase space M together with real functions $f : M \to \mathbb{R}$ that represent observable quantities. One acquires the knowledge through a pairing of system's state $\omega \in M$ and certain (elementary) propositions about the system via an assignment of a value $\omega(P)$ to the probability that the proposition P is true in a state ω. For example, P may state that an observable f has a value in $U = (a, b) \subseteq \mathbb{R}$ in a state ω. As ω is essentially a map $\{*\} \xrightarrow{\omega} M$ and the proposition P can be depicted as $M \xrightarrow{\chi_{f^{-1}(U)}} \{0, 1\}$, we get the following representation of the pairing

$$1 = \{*\} \xrightarrow{\omega} M \xrightarrow{\chi_{f^{-1}(U)}} \{0, 1\} = \Omega, \tag{2.17}$$

where 1 denotes the initial object (a singleton) and Ω is the subobject classifier in the topos **Set**. However, (2.17) *cannot* be naturally translated to quantum formalism in this form. Indeed, although it is tempting to adopt the formalism introduced in Section 2.1.1 and construct the pairing through states $\psi \in \mathcal{H}$ and propositions represented by closed subspaces of $\mathcal{H}$, the characteristics of quantum logic such as nondistributivity or lack of implication of $\mathcal{L}(\mathcal{H})$ pose a problem. Instead, the authors of [117] propose to shift the perspective and treat the topology $\mathcal{O}(M)$ as the primitive notion. Then, the proposition P can be represented by $1 \xrightarrow{f^{-1}(U)} \mathcal{O}(M)$. States are no longer elements of M; instead they are represented by subobjects $S_\omega = \{U \in \mathcal{O}(M) : \omega \in U\}$. Finally the pairing is reversed with respect to (2.17) as

$$1 \xrightarrow{P} \mathcal{O}(M) \xrightarrow{\chi_{S_\omega}} \Omega.$$

Recall that $\mathcal{O}(M)$ gives rise to the structure known as a frame and further, a locale (see Appendix B) and it is this localic perspective that gives an appropriate logical (although intuitionistic) framework for quantum mechanics.

With the most important facts related to topos theory in mind (see Appendix B), let us summarize what the covariant approach is. Let A be a unital (noncommutative in general)

C^*-algebra and $\mathcal{C}(A)$ be a family of all unital, commutative C^*-subagebras of A. This way, $\mathcal{C}(A)$ can be partially ordered by inclusion and turned into a posetal category. The starting point for Bohrification is a presheaf topos (see Example 19)

$$\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}.$$

As every topos carriers its own language and can be considered a "universe of mathematical discourse", we test how much will change as we move from external topos **Set** to the one related to "classical snapshots" — $\mathcal{T}(A)$. In a sense, we assign a certain "frame of reference" with each A, and we will find out, whether it is possible to turn the total algebra A into a commutative (i.e. classical) C^*-algebra. (All objects internal to $\mathcal{T}(A)$ will be underlined, to distinguish them from notions being described externally, from the point of view of **Set**). Consider now the forgetful functor $\underline{A} \in \mathrm{Ob}\left(\mathcal{T}(A)\right)$, which sends each $C \in \mathcal{C}(A)$ to its underlying set and each arrow $C \subseteq D$ to the inclusion arrow $C \hookrightarrow D$. It can be shown that that internally $\underline{A}$ becomes a commutative C^*-algebra over the internal set of complex numbers [81]. Furthermore, using Gelfand duality one constructs internally a locale (see Appendix B) $\underline{\Sigma}(\underline{A})$ representing the Gelfand spectrum of $\underline{A}$, which becomes a candidate for quantum phase space; the subobjects-opens $1 \to \mathcal{O}(\underline{\Sigma})$ represent elementary propositions and the observables $a \in A$ regain their classical form as locale maps $\delta(a) : \underline{\Sigma}(\underline{A}) \to \underline{\mathbb{IR}}$, where $\underline{\mathbb{IR}}$ is a poset of compact intervals in $\mathbb{R}$, ordered reversely by inclusion. In particular, one proves the category-theoretic version of Theorem 18 [116]

Theorem 9. Let $\mathcal{H}$ be a Hilbert space such that $\dim(\mathcal{H}) > 2$ and $A = \mathcal{B}(\mathcal{H})$. Then the locale $\underline{\Sigma}(\underline{A})$ has no points.

Thus the topos approach offers a different perspective on the fact that quantum observables do not have globally defined real values.

Now we turn to a specific type of a topoi, called *smooth* topoi, which serve as models for *synthetic differential geometry* (SDG). Synthetic differential geometry traces its origins to the work [105] and, in essence, is a formalization of the Leibnizian, limit-free approach to understanding differential geometry ([106] is an excellent introduction to the subject). It is kind of a folklore that differential calculus, since the very beginning, have been understood in a two-fold way: the one that operates on infinitesimals (infinitely small segments) advocated by Leibniz, and the other relying on the notion of a limit, which found its formal culmination in the work of Cauchy. (Without a doubt, the latter has found much more followers and it has become the primary description of the smooth arena in the mathematical world.) Basically, the most fundamental object of interest in SDG is the collection of (first-order) infinitesimals

$$\mathbf{D} = \{d \in \mathbf{R} \,|\, d^2 = 0\},$$

called the nilpotent infinitesimals. Then, the fundamental Kock–Lawvere axiom of SDG claims that all functions coincide with straight-line segment in the infinitesimal neighbourhood of 0, i.e. for every $f : \mathbf{D} \to \mathbf{R}$ there is a unique $b \in \mathbf{R}$ such that for all $d \in \mathbf{D}$ it holds

$$f(d) = f(0) + b \cdot d.$$

Observe that the object $\mathbf{R}$, despite aimed at representing real numbers, has to be different from the set standard real numbers $\mathbb{R}$. The property that sets $\mathbf{R}$ apart from $\mathbb{R}$ is its undecidability. Recall that a property P on a set X is decidable if

$$\forall x \in X \left(P(x) \vee \neg P(x)\right)$$

In particular, we call an object A decidable if

$$\forall x, y \in A\left(x = y \vee x \neq y\right).$$

It can be proved [106] that **R** is not decidable, thus the Kock–Lawvere axiom is not compatible with the law of excluded middle and SDG is inherently intuitionistic, which means it has to be formulated outside ZFC set theory to avoid such contradictions.

Now, a question may arise as to whether the reals **R** can assume the role of the "true" real numbers within some broader context. The answer is affirmative, and we will see that category theory gives a precise answer which is that of a smooth topos. However, it will come at a cost of introducing some other features, that change the way of conceiving smoothness in general. In particular, in Chapter 5 we will argue that the topos-theoretic local description of smooth structures may reveal new information on the problem of exotic smooth $\mathbb{R}^4$ (introduced in the Section 2.3). Here we refer mainly to [134],[11] and occasionally to [64].

Remark 16. One may wonder why approaching problems related to smooth manifolds via the category **Mfd** is not always sufficient, although seemingly most natural. The answer is that **Mfd** miss two important properties, namely being cartesian closed (in particular the exponential M^N does not have to be a manifold for manifolds M, N) and lacking finite inverse limits (pullbacks do not have to be manifolds either).

Recall that a smooth topos **T** is a topos that models SDG. If a category **Mfd** of smooth manifolds embeds fully and faithfully in such **T**, we call **T** well-adapted. Let us construct now a particular well-adapted topos, that will be of our interest mainly in Chapter 5.

Consider the category of finitely generated C^∞-rings (smooth rings) and C^∞-ring homomorphisms. Then the objects are rings of the form $C^\infty(\mathbb{R}^n)/I$ where I is an ideal in $C^\infty(\mathbb{R}^n)$ and for $B = C^\infty(\mathbb{R}^n)/J, A = C^\infty(\mathbb{R}^m)/I$, a morphism $\ell B \to \ell A$ is an equivalence class of smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that $f \in I \implies f \circ \phi \in J$ and $\phi \sim \phi'$ if $\phi - \phi' \in J$ for all $i = 1, \ldots, m$. We write $A \otimes_\infty B$ for a coproduct of smooth rings A, B.

The category opposite to finitely generated C^∞-rings is called the category of loci (plural for locus) **L**, so the objects of **L** are again finitely generated C^∞-rings; to avoid confusion, we will use the notation ℓA for an object in **L** where A is an C^∞-ring, hence the morphisms $\ell B \to \ell A$ are C^∞-morphisms $A \to B$.

One of the reasons we employ the category **L** is a reformulation of Theorem 14 in category-theoretic language [134]:

Theorem 10. The functor $s : \mathbf{Mfd} \to \mathbf{L}$, $M \mapsto \ell C^\infty(M)$ is a full and faithful.

We will employ now this characterization directly; given any $M \in \mathrm{Ob}(\mathbf{Mfd})$, there exists $n \in \mathbb{N}$ and an ideal I in $C^\infty(\mathbb{R}^n)$ such that

- $s(M) = C^\infty(M) = C^\infty(\mathbb{R}^n)/I$,
- $s(\mathbb{R}) = C^\infty(\mathbb{R})$
- $s(M \times N) = s(M) \otimes_\infty s(N)$,
- $s(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}) \otimes_\infty \ldots \otimes_\infty C^\infty(\mathbb{R}) = s(\mathbb{R}) \otimes_\infty \ldots \otimes_\infty s(\mathbb{R})$

Observe that already the image of $\mathbb{R}$ under s gives a more complicated object $R \equiv C^\infty(\mathbb{R})$. If we interpret R as the "reals inside" **L**, we see $\mathbb{R}$ embedded as constant functions. All the other smooth maps are "the new reals".

Remark 17. Unfortunately **L** is not cartesian closed. However, we will find that the category $\mathbf{Set}^{\mathbf{L}^{op}}$ enjoys demanded properties.

We embed now the category $\mathbf{L}$ in $\mathbf{Set}^{\mathbf{L}^{op}}$ through Yoneda embedding:

$$Y : \mathbf{L} \to \mathbf{Set}^{\mathbf{L}^{op}}, \quad Y(\ell A) = \mathbf{L}(-, \ell A)$$

and composing Y with s we obtain the functor embedding (denoted again by s)

$$s : \mathbf{Mfd} \to \mathbf{L} \to \mathbf{Set}^{\mathbf{L}^{op}}, \quad M \mapsto \underline{M} = \mathbf{L}(-, C^\infty(M)),$$

which is again full and faithfull. Importantly, $\mathbf{Set}^{\mathbf{L}^{op}}$ as a presheaf category is a topos (see Appendix B) and therefore it is cartesian closed.

Let $\mathbf{B}$ be a site defined by $\mathbf{L}$ with a Grothendieck topology. Let $\mathcal{B}$ denote the topos of sheaves over a site $\mathbf{B}$. One can show that all "nice" properties of $\mathbf{Set}^{\mathbf{L}^{op}}$ are preserved:

Theorem 11. There is a full and faithful embedding $s : \mathbf{Mfd} \to \mathcal{B}$. The Yoneda embedding $\mathbf{L} \to \mathbf{Set}^{\mathbf{L}^{op}}$ factors through $\mathcal{B} \subset \mathbf{Set}^{\mathbf{B}^{op}}$.

We finally obtain $\mathrm{Ob}(\mathbf{Mfd}) \ni M \mapsto s(M) = \mathbf{B}(-, C^\infty(M)) \in \mathrm{Ob}(\mathcal{B})$. Since $\mathcal{B}$ is a topos, set-theoretic operations and constructions are valid as long as they are constructive, i.e. the logic is intuitionistic and one does not refer to the law of excluded middle nor to the axiom of choice. Let us describe the topos $\mathcal{B}$ in more detail.

First of all, we go back and forth between $\mathbf{Set}$ and $\mathcal{B}$ with the help of a pair of adjoint functors $\Delta \dashv \Gamma$:

$$\mathbf{Mfd} \xrightarrow{\ s\ } \mathcal{B} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set} \tag{2.18}$$

The functor $\Delta : \mathbf{Set} \to \mathcal{B}$ is a constant sheaf functor (the sheafification of a constant presheaf), given by $\Delta(X)(\ell A) = X$ for every $\ell A \in \mathbf{L}$. Thus Δ defines the embedding of "constant sets" in the universe $\mathcal{B}$ of "variable sets". The functor $\Gamma : \mathcal{B} \to \mathbf{Set}$ is the so-called global section functor, given by $\Gamma(F) = F(1)$, where $1 = s(\{*\}) = \ell C^\infty(\mathbb{R})/x$ is the one-point locus. Note that for all manifolds $M, N \in \mathrm{Ob}(\mathbf{Mfd})$ it holds

$$\Gamma(s(M)) \overset{\text{diff}}{\cong} M, \quad \Gamma\left(s(N)^{s(M)}\right) \overset{\text{diff}}{\cong} C^\infty(M, N)$$

For example, the object of *smooth real numbers* $R_{\mathcal{B}}$ in $\mathcal{B}$ is canonically defined by

$$R_{\mathcal{B}} \equiv R = \mathbf{B}(-, C^\infty(\mathbb{R})).$$

Secondly, the way $\mathcal{B}$ is constructed leads to problems with understanding what is the "appropriate" object of natural numbers in $\mathcal{B}$. As a Grothendieck topos, $\mathcal{B}$ contains a natural numbers object $\underline{\mathbb{N}} = \Delta(\mathbb{N})$, i.e. the sheaf associated with a constant presheaf $\mathbf{L}^{op} \to \mathbf{Set}$ given by $\ell A \mapsto \mathbb{N}$. The problem with $\Delta(\mathbb{N})$ is that many plausible properties, such as the compactness of $[0, 1] \subset R$, Archimedean property of R etc. are not satisfied with respect to $\Delta(\mathbb{N})$. Thankfully, there is another candidate for natural numbers object: the image of $\mathbb{N}$ under the full and faithfull embedding $N = s(\mathbb{N}) = \ell C^\infty(\mathbb{N})$, called the *smooth natural numbers*. Interestingly, R can be shown to contain non-standard elements: infinite natural numbers $N \setminus \mathbb{N}$ and the object of infinitesimals $\square$ (which further splits into invertible $\mathbb{I}$

and nilpotent Δ ones):

$$\square = \{x \in R : \forall n \in \mathbb{N} \left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right)\},$$
$$\mathbb{I} = \square \cap U(R), \text{ where } U(R) \text{ consists of invertible reals},$$
$$\Delta = \{x \in R : \forall n \in N \left(-\frac{1}{n+1} < x < \frac{1}{n+1}\right)\}.$$

By systematically replacing canonical $\Delta(\mathbb{N})$ with smooth N, the demanded properties, such as preserving compactness via $s : \mathbf{Mfd} \to \mathcal{B}$ or the Archimedean property of R, are restored.

One of the remarkable properties of $\mathcal{B}$ is the way distributions are handled there. Recall that in the category **Set** of sets and functions, the concept of a distribution was brought to life in order to deal with peculiarities like the delta "function"

$$\int f(x)\delta(x)\mathrm{d}x = f(0). \tag{2.19}$$

The fact that is easy to verify, yet usually swept under the rug, is that the function $\delta(x)$ above does not exist in **Set** at all, and it serves as a mere imagination of a map with an infinite spike located at $x = 0$. The rigorous framework of introducing such an object is that of a distribution theory. Let $C_c^\infty(\mathbb{R}^n)$ denote a vector space of smooth functions (test functions) with a compact support in $\mathbb{R}$, what leads to a generalization of a derivative:

$$\int (\mathrm{D}^\alpha f)(x)\phi(x)\mathrm{d}x = (-1)^{|\alpha|} \int f(x)(\mathrm{D}^\alpha \phi)(x)\mathrm{d}x \tag{2.20}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \dots \mathrm{D}_4^{\alpha_4}$ with $\mathrm{D}_j = \frac{\partial}{\partial x_j}$. Observe that the integral on the right-hand side of (2.20) defines a linear functional on $C_c^\infty(\mathbb{R}^n)$ and does not depend on differentiability of f. Let $D'(\mathbb{R})$ denote the set of distributions, i.e. continuous linear functionals on $C_c^\infty(\mathbb{R}^n)$. Consequently, (2.19) gets a firm basis as a distribution $\delta : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ given by

$$\delta(f) = f(0).$$

As stated earlier, δ is a good example of a distribution that is not regular, i.e. for which an integral representation does not exist. On the other hand, every locally integrable function g gives rise to a regular distribution via

$$T_g(f) = \int f(x)g(x)\mathrm{d}x.$$

Let us present the above inside $\mathcal{B}$. Recall that in order not to fall into problematic issues, we decided to replace the canonical NNO, i.e. the constant sheaf $\underline{\mathbb{N}}$ by the object N of smooth natural numbers. This is also crucial with respect to handle distributions in $\mathcal{B}$. To proceed, we need the object R_{acc} of *accessible reals*:

$$R_{\mathrm{acc}} = \{x \in R : \exists n \in \mathbb{N} \, (-n < x < n)\}.$$

Then $f : R^n \to R$ is called accessible if for every multi-index α and every $x \in R_{\mathrm{acc}}$ we have $\mathrm{D}^\alpha f(x) \in R_{\mathrm{acc}}$. Furthermore, define $(R^{R^n})_a$ to be composed of functions with *accessible support*:

$$(R^{R^n})_a = \{f : R^n \to R : \exists m \in \mathbb{N} \, \forall x \in R^n \left(x \in [-m, m] \vee f(x) = 0\right)\}.$$

Here, the set F_n of *test functions* consists of accessible functions with accessible supports. Then *a distribution* on R^n is an R-linear map $\mu : (R^{R^n})_a \to R$ which is "infinitesimally" continuous, i.e.

$$\forall x \in R^n \, \forall \alpha \left(D^\alpha f(x) \simeq 0 \implies \mu(f) \simeq 0 \right). \tag{2.21}$$

Here $a \simeq b$ means that $a - b \in \Box$ (i.e. the reals a, b are indistinguishable "up to infinitesimals").

Finally we arrive at two most important properties of distributions in $\mathcal{B}$. Firstly, we introduce *predistributions* — these are functions $\varphi : R^n_{\text{acc}} \to R$ such that $\Delta_\varphi : F_n \to R$ defined by

$$\Delta_\varphi(f) = \int_{[-A,A]^n} \varphi f \quad \text{with } A \text{ such that } f(x) = 0 \text{ for } x \notin [-A, A]^n$$

is infinitesimally continuous in the sense of (2.21). We see that predistributions are the functions giving rise to what we know as *regular distributions*, i.e. those representable by locally integrable functions in the standard approach [155]. Amazingly, all internal distributions in $\mathcal{B}$ are of this type due to the following fact [134]:

Theorem 12. For every distribution $\mu : R^n \to R$ there exists a predistribution $\mu_0 : R^n_{\text{acc}} \to R$ such that for all $f \in F_n$ it holds

$$\mu(f) \simeq \Delta_{\mu_0}(f) = \int \mu_0 f.$$

Secondly, it can be shown that there is a direct relation between distributions internal to $\mathcal{B}$ and external distributions living in **Set** [134]:

Theorem 13. The functor $\Gamma : \mathcal{B} \to$ **Set** induces a bijection between internal distributions $\mu : F_n \to R$ and external distributions $\Gamma(\mu) : C^\infty_c(\mathbb{R}^n) \to \mathbb{R})$, as well as between internal distributions with compact support, i.e. R-linear maps $\mu : R^{R^n} \to R$ and external distributions with compact support $\Gamma(\mu) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$.

The above means that we can go back and forth between distributions internal and external with respect to $\mathcal{B}$; in particular, every external distributions becomes a regular distribution internally in $\mathcal{B}$. In the next section we will get familiar with the existence of exotic smooth structures, especially in dimension 4. As we will see, such manifolds could be in principle analyzed given the characteristics of functions defined on them. As we have already seen, Basel topos $\mathcal{B}$ gives quite a new picture on the way distributions are represented and general functions are differentiated, what gives promising opportunities to shed new light on the puzzle of exotic smoothness.

2.3 Exotic smooth structures

We have already discussed that physics has a rich history of challenging established assumptions. In particular, the ideas related to non-cartesian mixing of space and time, the concept of curvature or non-trivial spacetime topology have had profound impact scientific research. In this section, we will briefly go through the history of questioning yet another foundational assumption: the standard differential structure of a spacetime.

In general, the object $\mathbb{R}$ of real numbers may appear in various ways as:

- a set of values that physical observables attain,
- a quantitity parametrizing an arrow of time,

- a real numbers object in any topos with NNO,
- the Cantor set 2^ω of all subsets of natural numbers,
- a unique complete ordered field and further — real numbers $\mathbb{R}_M$ formalized in ZFC models.

It is possible that each of above contexts has its own natural approach, with a prominent role in describing the natural world. However, all of them refer to $\mathbb{R}$ more or less as a set, sometimes with a bit of an additional structure, such as ordering, or topology. At the same time, it is rather evident that to study dynamics in the general setting, including spacetime phenomena, more involved frameworks are required. In physics, spaces of higher dimensions like the (apparently) four-dimensional spacetime, $6N$-dimensional phase space of N classical particles or infinite-dimensional Hilbert spaces, are a constant theme. Even for configuration spaces it is occasionally necessary to formulate problems on manifolds that do not resemble an Euclidean one, at least globally. For example, consider the configuration space of a two-dimensional double pendulum: two independent circles sweep out a surface of torus $S^1 \times S^1$, which certainly does not look like a plane when viewed globally. Furthermore, to account for phenomena that occur in a local and continuous manner (or at least to model them, if we do not believe in such fundamental locality and smoothness), some form of dynamics has to be incorporated. Moreover, to deal with instantaneous rates of physical quantities like velocity, it becomes necessary to discuss limits (topological structure) and derivatives (differential structure) in the first place. In order to reach this point and handle the most general scenarios, one follows the usual and accepted path through defining topological and differential structure on a given set. Finally, to develop the full theory e.g. of fields, one introduces geometry, mostly in the language of bundles. Therefore, the usual path goes like that [6]:

$$\text{point set} \to \text{topology} \to \text{smoothness} \to \text{bundles} \to \text{geometry} \tag{2.22}$$

While it is tempting to delve deeper into the story and further stages, such as metrics, connections, gauge theory etc., let us pause here and focus on the middle part of (2.22)), i.e. the topology-smoothness aspect. For the sake of clarity, we will restrict our discussion to infinite differentiability in the context of a smooth structure. In the following we recall several related definitions and facts. An n-dimensional, smooth manifold $(M, \mathcal{A})$ consists of n-dimensional, Hausdorff topological manifold M and a maximal atlas $\mathcal{A} = (\{U \,|\, U \subseteq M \text{ — open}\}, \{\phi_U\})$, where charts $\phi_U : U \to \phi(U) \subseteq \mathbb{R}^n$ are homeomorphisms and all compositions (transition functions) $\phi_U^{-1} \circ \phi_V : \mathbb{R}^n \to \mathbb{R}^n$ are smooth. A map between manifolds is called *smooth* whenever it is smooth as expressed by charts and such definition ensures that smoothness is well-defined. In particular, a map $f : M \to \mathbb{R}$ is smooth at $p \in M$ whenever all coordinates $\{\phi_U^i\}$ are smooth at $p \in M$ for some chart (U, ϕ_U). Two manifolds are diffeomorphic if there exists a smooth bijection (diffeomorphism) between them. One has to be aware of the difference between nondiffeomorphic smooth structures and those that are just different. For instance, in general relativity a diffeomorphism is interpreted as a re-coordinatization of a manifold M, i.e. every coordinate change is connected to some diffeomorphism applied to M. Consequently, diffeomorphic manifolds are precisely those that represent the same physical content from the perspective of general relativity.

Example 5. Let M be a manifold $\mathbb{R}$ with the smooth structure defined by a global chart $\phi(p) = p \in \mathbb{R}$ for $p \in M$. Now let us design a new structure M' given by $\mathbb{R}$ with a different chart $\psi(p) = p^{\frac{1}{3}}$. Structures generated by (M, ϕ) and (M', ψ) are clearly different, since the transition function

$$(\psi \circ \phi^{-1})(p) = p^{\frac{1}{3}}$$

is not smooth. However, it is easy to observe that there is a homeomorphism $f : p \mapsto p^3 \in \mathbb{R}$ that is smooth by

$$(\psi \circ f \circ \phi^{-1})(p) = (\psi \circ f)(p) = \psi(p^3) = p.$$

We will also use a description of a smooth manifold M by the respective ring $C^\infty(M)$ of smooth maps $M \to \mathbb{R}$. Importantly, it can be proved in several ways that nothing is lost here (see Lemma 11) and also it holds [71]

Theorem 14. Let M, N be separated smooth manifolds, whose topologies have countable bases. There is a natural bijection

$$\mathrm{Hom}(M, N) \simeq \mathrm{Hom}(C^\infty(N), C^\infty(M)),$$

where on the left-hand side we have smooth maps and on the right-hand side there are $\mathbb{R}$-algebra homomorphisms.

Remark 18. We will be interested mainly in $\mathbb{R}^4$, thus it is worth noting that the topology on $\mathbb{R}^4$ comes from the standard product topology of $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. More importantly, the *standard smooth structure* on Euclidean $\mathbb{R}^n$ is defined by the atlas that consists of a global identity chart $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. In the case of $n = 4$, it is the unique smooth structure such that the product $\mathbb{R}^3 \times \mathbb{R}$ is also smooth (more generally, it can be shown that if M is a smooth, contractible 3-manifold, then $M \times \mathbb{R}$ is PL-isomorphic to $\mathbb{R}^4$ and this is sufficient for $M \times \mathbb{R}$ to be diffeomorphic to standard $\mathbb{R}^4$ [138].

Suppose someone hands us a topological manifold M and asks whether there exists a unique smooth structure on M (granted there is at least one). Put differently, are two homeomorphic smooth manifolds necessarily diffeomorphic? This question was initially explored in the context of spheres; the first constructive results were obtained by Milnor [133] revealing smooth structures on S^7, nondiffeomorphic to the "standard" one. Subsequent efforts led to the famous theorem by Kervaire and Milnor [100], which states that there are only finitely many exotic smooth structures on S^n for all $n \geq 5$. The cases of $n = 2$ and $n = 3$ had already been resolved by Radó [151] and Moise [135], respectively. However, the case of S^4, called the smooth Poincaré conjecture, remains an open problem — it is still unknown whether a four-dimensional sphere admits an exotic smooth structure (cf. e.g. [66]). Meanwhile, the situation for $\mathbb{R}^n$ is paradoxical, being both manifest and challenging at the same time. Indeed, there is essentially only one differential structure for each $\mathbb{R}^n$ with $n \neq 4$; cases $n = 1, 2, 3$ are relatively straightforward [136], while uniqueness for $n > 5$ is due to Stallings [167] and follows in particular from the famous h-cobordism theorem by Smale [165]. Therefore, it was quite natural to conjecture the uniqueness of smooth structure of $\mathbb{R}^n$ for all $n \in \mathbb{N}$. Unexpectedly, the situation changes significantly for $n = 4$: not only there exist smooth structures not diffeomorphic to the standard one, but there are uncountably many of them [67, 53] and we call them *exotic* smooth (or fake) $\mathbb{R}^4$. Furthermore, exotic smooth $\mathbb{R}^4$'s split into two classes: *small* exotic $\mathbb{R}^4$ that can be embedded in the standard $\mathbb{R}^4$ as open subsets, and *large* exotic $\mathbb{R}^4$ for which such an embedding does not exist. Interestingly, each class contains uncountably many different smooth structures as well. Essentially, small exotic $\mathbb{R}^4$ arise due to failure of smooth h-cobordism and large exotic $\mathbb{R}^4$ are the consequence of a failure of surgery theorem in the smooth case. Since small exotic $\mathbb{R}^4$ are of main interest here, we give a brief account on their origin and properties.

Firstly, recall that in dimensions greater than 4, the h-cobordism theorem states that a smooth equivalence between compact and simply-connected manifolds M, N is controlled by the h-cobordism W between them [159]. It means that whenever there exists an $(n+1)$-dimensional W such that $\partial W = \bar{M} \cup N$ (orientation of M reversed) and W is homotopically equivalent to $M \times [0,1]$, then also smoothly $W \simeq M \times [0,1]$ and in particular M, N are diffeomorphic. The crucial technique that established the theorem for $n \geq 5$, called the Whitney trick, relies on embedding an open 2-handle (a Whitney disk) in order to eliminate the pairs of intersection points inside the cobordism. However, in 4 dimensions the method breaks down since the immersed disk may introduce self-intersections as well. What Casson proposed instead was the *infinite* procedure of iterative disk immersion, now known as *Casson handle*: a union over a tower of disks with well-defined levels, where disks of a given stage cancel self-intersection from the preceding stage, but introduce new ones [45]. Surprisingly, any such Casson handle is topologically just $D^2 \times \mathbb{R}^2$, what resulted also in a topological h-cobordism theorem in $n = 4$ [67]. Still, the theorem does not hold smoothly. It is worth noting that Casson handles can be reconstructed from infinite, $(+,-)$-signed tree: at every level, vertices and signs on branches correspond to self-intersections and their signs, respectively, and the number of branches attached to given vertex is equal to the number of self-intersections on the next level, and there are sufficiently many Casson handles to cover all trees in such a way [94]. Now, two "simplest", linear Casson handles with one self-intersection of a constant sign (denoted $\mathrm{CH}_+ \equiv (+,+,\ldots)$, $\mathrm{CH}_- \equiv (-,-,\ldots)$, respectively) at each level, can be shown to be exotic [32]. At the same time, all Casson handles embeddable in either CH_+ or CH_- are exotic. While it is still an open question whether a linear Casson handle with arbitrary signs is exotic, an affirmative answer would suffice for all Casson handles to be exotic as well [32]. Note that such a general linear $(+,+,-,+,\ldots)$ Casson handles can be represented surjectively by real numbers given their a binary expansions (one needs only to replace $+,-$ with 0, 1, respectively), hence equivalently to binary tree, and we will return to this observation in Chapter 7. Now one might wonder whether it is possible to "localize" exotic smoothness. The answer is surprising indeed, as given a smooth h-cobordism between 4-manifolds M, N, inside W there exists a compact contractible sub-h-cobordism K between compact contractible submanifolds $A \subset M$, $B \subset N$ (thus $\partial K = A \sqcup B$) such that W is smoothly trivial *outside* K, i.e.

$$W \setminus \mathrm{Int}K \simeq (M \setminus \mathrm{Int}A) \times [0,1].$$

In this case A determines the smooth structure of M; we call such A the *Akbulut cork*, described originally in [4]. The term "cork" comes from the fact that one obtains different smooth structures by cutting A from M and regluing A by an involution of ∂A. Importantly, the boundary of such A is always a homology sphere [67] and moreover, Akbulut corks are always surrounded by small exotic $\mathbb{R}^4$ [159]. All of the above will be crucial in demonstrating in Chapter 4 that appropriate exotic smoothness produces realistic cosmological parameters.

Remark 19. One might obviously debate the reasons behind the complexity of dimension 4 when it comes to smooth structures on $\mathbb{R}^n$. As we have discussed, from the "technical" point of view, the case is more or less manifest. But if we ask why the things are the way they are, there could be two interconnected, "anthropic" explanations:

- we cannot formulate hard questions about smoothness of dimensions higher than 4; instead, we "project" our inquiries based on the low-dimensional questions instead,

- our mathematics reflects the human way of thinking, which is inherently four-dimensional (even if higher dimensions do exist fundamentally, we seem to be bounded to effective, four-dimensional perception), and this inherent limitation affects the approach to higher-dimensional problems.

While the first point seems reasonable, we object to the second one: it merely does not provide a convincing explanation for why dimension 4 should be more complicated than other dimensions. Nonetheless, finally we would like to refer another opinion [159]: "dimension 4 is large enough to allow strange things to happen, but too small to enable one to undo them".

Let us now discuss the fascinating topic of the potential physical implications of just mentioned results on the exotic smoothness of $\mathbb{R}^4$. In the remaining part of this Section we will recall some applications of exotic smoothness in physics that have appeared in the literature so far. It will also give a proper background for the results in Chapter 4. As discussed in [37], it may seem counter-intuitive that exotic smooth structures have received relatively little attention in the physics community, given the multitude of motivations they incorporated from physics, such as Yang–Mills and string theory. One reason for such a situation is certainly the technical difficulties related to the subject, such as deficiency of explicit chart descriptions; inherently infinite constructions, which we briefly discussed in the previous section, are also not encouraging. Secondly, there is a strong belief that differential structure is inherently untestable, partly due to the existence of an unknown (if any) fundamental length. Consequently, one could argue that the choice of smoothness does not matter as long as it serves merely as a convenient modelling tool. Moreover, even if we assume spacetime to be fundamentally smooth, the effects of inequivalent differential structures would not be locally detectable, as all smooth $\mathbb{R}^4$ appear identical from the perspective of local charts (after all, the local resemblance to $\mathbb{R}^n$ of *any* smooth manifold is a central point in the definition of a smooth manifold). In practice, it is a standard approach in physics to deal with homeomorphic manifolds *as they are* also diffeomorphic, i.e. we do not concern ourselves which smooth atlas we actually use (neglecting somehow the mathematical reservations).

Remark 20. We refer to the above comments occasionally. Briefly, it can be argued that:

- the relationship between the notions of discrete and continuous (smooth) can be understood in both directions: the set of reals $\mathbb{R}$ is not only a refinement of natural numbers $\mathbb{N}$ through integers $\mathbb{Z}$ and rationals $\mathbb{Q}$, but e.g. internal natural numbers N in a smooth topos arise as a result of sheaf construction on the given smooth algebra (see Section 2.2.3), therefore N can be also considered as built "on top" of $\mathbb{R}$. In such a perspective, a perfect smoothness of our spacetime may coexist with a fundamental length scale in a nontrivial way,

- the fact that nondiffeomorphic structures cannot be distinguished locally does not exclude the possibility that smoothness might have an impact on physics. In particular, prior to the discovery of exotic smoothness, it was widely believed that the only global effects in physics were related to topology alone (cf. e.g. Aharonov–Bohm effect [161]). As exotic smoothness of $\mathbb{R}^4$ is inherently global, its potential role in topology-like consequences in physics has to be investigated. It is important to note that diffeomorphisms represent coordinate transformations and therefore nondiffeomorphic smooth structures imply inequivalent descriptions of the same spacetime.

Due to the inherently global character of exotic smoothness, physicists began to explore its implications a few years after mathematicians made groundbreaking discoveries in this area. This exploration dates back to the work of Witten [174], who demonstrated that 11-dimensional exotic spheres could be interpreted as gravitational instantons, along with the results of Baadhio et al. [18]. Around the same time, Brans started the series of papers that significantly contributed to our understanding of the potential impact of exotic smoothness on the description of spacetime. Initially one could show that, although coordinate maps $\{x_i\}$ on a topological $\mathbb{R}^4$ cannot be *globally* smooth with respect to exotic smooth structure, there always exists some neighbourhood such that $\{x_i\}$ are *locally* smooth inside this neighbourhood [35]. Moreover, the "exoticness" of $\mathbb{R}^4$ can be "spatially localized" in the following sense [36].

Theorem 15. There exists smooth manifolds which are homeomorphic but not diffeomorphic to $\mathbb{R}^4$ and for which the global topological coordinates (t, x, y, z) are smooth for $x^2 + y^2 + z^2 \geq a^2 > 0$, but not globally. Smooth metrics exists for which the boundary of this region is timelike, so that the exoticness is spatially confined.

Based on this result, Brans proposed a famous conjecture, now known as Brans conjecture: *Localized exoticness can act as a source for gravitational field, just as ordinary matter does* [6]. Eventually, the conjecture was proven by Asselmeyer-Maluga [5] for a compact case and by Sladkowski [164] for a non-compact case. Exotic smooth structures were later used e.g. in Euclidean Quantum Gravity [56], string theory [9], black holes [7], cosmology [12], geometric origin of matter [15]. Exotic smooth structures on $\mathbb{R}^4$ have been studied numerously with the use of model theory and category theory (e.g. [110], [108], [11], [14]) and we frequently refer to these works through the thesis. Recently, exotic $\mathbb{R}^4$ have been shown to represent gravitational instantons violating Strong Cosmic Censorship Conjecture [62].

Chapter 3

The structure of $\mathcal{L}(\mathcal{H})$

In Chapter 2 we went through a rather heavy tour over several topics, spanning from quantum mechanics through smooth manifolds to a highly abstract categories and model theory. In the current chapter we shall introduce some details concerning the structure of the lattice $\mathcal{L}(\mathcal{H})$, with the particular focus on its Boolean subalgebras. We have already got familiar with the evolution of ideas that had drawn inspiration from the thoughts primarily attributed to Bohr. Thus we have encountered the posetal categories of commutative substructures, both within the context of von Neumann and C^*-algebra $\mathcal{B}(\mathcal{H})$ associated with a quantum system. We propose to shift the perspective slightly and look closer at another setup, primarily related to logic. Namely, instead of substructures of $\mathcal{B}(\mathcal{H})$, we will deal with an orthomodular lattice $\mathcal{L}(\mathcal{H})$ and its Boolean subalgebras. The first advantage of such an approach is that $\mathcal{L}(\mathcal{H})$ is glued "nicely" from the structure BSub($\mathcal{L}(\mathcal{H})$) of Boolean subalgebras in the language of category theory; this is rather obscure in the setup of general C^*- or von Neumann algebras and their commutative subalgebras. The second advantage is not visible immediately, although we will see later on that it allows to understand the "classical snapshots" (Boolean subalgebras) as directly giving rise to "mathematical universes" (Boolean topoi), therefore making the connection with category- and model-theoretic description introduced in Chapter 2. These will serve as a unifying thread, running through subsequent chapters. Furthermore, subalgebras of $\mathcal{L}(\mathcal{H})$ give rise to the parametrization of smooth spacetime manifold, involving also a category-theoretic language; eventually we arrive at the problem of quantum-mechanical randomness, discussed again within the structure of $\mathcal{L}(\mathcal{H})$. Surprisingly, another reappearing structure is an exotic smoothness that seems to connect above with the smooth topos $\mathcal{B}$ as well. For details concerning lattices and Boolean algebras we refer the reader to Appendix A. We start with some well-known concepts and propositions.

3.1 Boolean subalgebras of $\mathcal{L}(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space of a quantum system; as already justified, we will assume $\mathcal{H}$ to be separable. Again, $\mathcal{B}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ are the C^*-algebra of bounded operators and the lattice of projections on $\mathcal{H}$, respectively. Recall that every projection operator $p \in \mathcal{L}(\mathcal{H})$ represents a "yes-no" question one might ask about the system, and the most basic questions are projections on one-dimensional subspaces of $\mathcal{H}$ — these are the atoms of $\mathcal{L}(\mathcal{H})$.

Remark 21. The complete, orthomodular lattice of projections $\mathcal{L}(\mathcal{H})$ always exists, provided $\mathcal{B}(\mathcal{H})$ on a separable $\mathcal{H}$. However, it might fail in the more general setting of unital C^*-algebras. In particular, despite each abstract C^*-algebra can be realised as a subalgebra of the concrete C^*-algebra $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (see Theorem 1), the specific space $\mathcal{H}$ mail fail to be separable. This is precisely the case of a Calkin algebra $\mathcal{C}(\mathcal{H})$ which will be of interest later on, due to peculiar relationships with ZFC set theory. In such case, $\mathcal{L}(\mathcal{C}(\mathcal{H}))$ might not even be a lattice [63].

Provided the background in Chapter 2, it is clear that quantum mechanics is algebraically related to noncommutativity of observables. In fact, this relation is exact, which follows from [160]

Theorem 16. A collection of observables is simultaneously observable if and only if it is commutative.

Now that we already have observed that quantum nature is reflected somehow in the logical structure of $\mathcal{L}(\mathcal{H})$, we still do have an access to "classical" substructures in $\mathcal{L}(\mathcal{H})$. These are Boolean subalgebras of $\mathcal{L}(\mathcal{H})$, i.e. subsets $B \subseteq \mathcal{L}(\mathcal{H})$ that are Boolean algebras on their own. Due to Lemma 1, $\mathcal{L}(\mathcal{H})$ cannot be Boolean whenever $\dim(\mathcal{H}) \geq 2$. By Lemma 2, it can also be viewed as yet another reflection of the simple fact, that there is no representation of a nonabelian C^*-algebra as a commutative subalgebra of $\mathcal{B}(\mathcal{H})$. Thus, one may use the strategy similar to Bohrification, i.e. to replace the noncommutative structure by its commutative substructures. The first natural choice is to decompose $\mathcal{L}(\mathcal{H})$ into its Boolean subalgebras.

We introduce $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ to be the set of all Boolean subalgebras of $\mathcal{L}(\mathcal{H})$, partially ordered by inclusion, i.e.

$$B_1 \leq B_2 \iff B_1 \subseteq B_2$$

for $B_1, B_2 \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$. We ensure that every $\mathcal{L}(\mathcal{H})$ can be covered with its Boolean subalgebras.

Lemma 4. Every $p \in \mathcal{L}(\mathcal{H})$ is contained in at least one Boolean subalgebra.

Proof. We propose the most trivial choice, i.e. given $B_p = \{0, p, \neg p, 1\}$, $p \in \mathcal{L}(\mathcal{H})$. □

It is then easy to see that

$$\mathcal{L}(\mathcal{H}) = \bigcup_{p \in \mathcal{L}(\mathcal{H})} B_p.$$

While formally true, the claim may be of little practical importance, as algebras of the form B_p provide a rather limited amount of information about the system. Specifically, one would like to enlarge these as much as possible, and there appears a natural question, whether there exists a notion of a maximal Boolean subalgebra that gives as much information as possible, together with a "classical" picture. This is indeed possible by the following

Lemma 5. Every Boolean subalgebra of $\mathcal{L}(\mathcal{H})$ can be extended to a maximal one.

Proof. It is known that $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ is a meet-semilattice (i.e. for each non-empty finite subset the greatest lower bound exists in $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$) and every chain of its elements is well-ordered [78]. By Zorn's lemma, every $B \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ is contained in $B' \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ that is maximal. □

Therefore every $B \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ (thus every $p \in \mathcal{L}(\mathcal{H})$) can be extended to at least one maximal Boolean subalgebra. Such maximal Boolean subalgebras will be further called *blocks* and they are indeed maximal with respect to the partial order in $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$. It is also easy to prove that all such blocks are automatically complete as Boolean algebras, since every block is subcomplete [153].

It is known that no information is lost when we pass from $\mathcal{L}(\mathcal{H})$ to a poset $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$, as orthomodular lattices are determined up to an isomorphism by their Boolean subalgebras via [78]

Theorem 17. Let $\mathcal{L}, \mathcal{M}$ be orthomodular lattices and $\phi : \mathrm{BSub}(\mathcal{L}) \to \mathrm{BSub}(\mathcal{M})$ be an order isomorphism. Then, there is a lattice isomorphism $\phi^* : \mathcal{L} \to \mathcal{M}$ with $\phi^*[B] = \phi(B)$ for every Boolean subalgebra $B \subseteq \mathcal{L}$, and ϕ^* is unique provided $\mathcal{L}$ has no four-element block.

One of the apparent weaknesses of studying $\mathcal{L}(\mathcal{H})$ instead of full $\mathcal{B}(\mathcal{H})$ is that $\mathcal{L}(\mathcal{H})$ contains only particular observables-elements of $\mathcal{B}(\mathcal{H})$ and lacks some self-adjoint operators. Yet, it is important to include these in the discussion of Bohr's doctrine. In order to do that, we will use a variant of the spectral theorem (Theorem 3) stating that every self-adjoint operator can be decomposed into appropriate family of projections. We refer here to the particular form of spectral theorem, that emphasizes the role of Boolean subalgebras of $\mathcal{L}(\mathcal{H})$ in recovering sets of commensurable observables [170].

Theorem 18. For every family $\{A_i\}_{i\in I}$ of self-adjoint pairwise commuting operators, there exists a complete Boolean algebra of projections B, such that given the spectral decompositions of each A_i

$$A_i = \int \lambda \mathrm{d}E^i_\lambda, \tag{3.1}$$

it holds that $\forall i \in I \left(\mathrm{d}E^i_\lambda \in B\right)$.

In particular, for every self-adjoint $A \in \mathcal{B}(\mathcal{H})$ one finds at least one complete Boolean algebra of projections such that A can be recovered from, in the sense of (3.1), which is just a generalization of a well-known formula in finite dimensions

$$A = \sum \lambda_i P_i$$

decomposing A into projections P_i on eigenspaces of A. We will say that a self-adjoint A is *contained* in a Boolean subalgebra B whenever $\{\mathrm{d}E^i_\lambda\} \subseteq B$ for a spectral resolution $\{\mathrm{d}E^i_\lambda\}$ of A. Due to Lemma 5 we conclude

Corollary 1. Every self-adjoint A is contained in at least one block.

Therefore, we can base the further discussion on (maximal) Boolean subalgebras of $\mathcal{L}(\mathcal{H})$.

Remark 22. There is a similar result for general self-adjoint operators: every such operator can be "extended" to the maximal set of commuting operators in $\mathcal{B}(\mathcal{H})$.

It is instructive to see particular examples of atomic and atomless Boolean algebras in the context of spectral resolutions of quantum-mechanical operators.

Example 6. Consider $\mathcal{H} = \mathbb{C}^2$ with a basis $\{|0\rangle, |1\rangle\}$ (see Example 1). It is easy to observe that given projections

$$p_0 = |0\rangle\langle 0|, \quad p_1 = |1\rangle\langle 1| = 1 - p_0$$

the Boolean algebra $(0, p_0, p_1, 1)$ is atomic.

Example 7. Let $\mathcal{H} = L^2(\mathbb{R})$ and define the position operator Q by

$$Q(\psi)(x) = x\psi(x).$$

Let $B_Q \subseteq \mathcal{L}(\mathcal{H})$ be the block containing Q. It is easy to see that B_Q has to be atomless, since Q is multiplicity-free, hence it defines a maximally commuting system of observables on $\mathcal{H}$ [33]. Therefore every $A \in \mathcal{B}(\mathcal{H})$ commuting with Q has to be of the form $f(Q)$, where f is measurable. Consequently, the spectral resolution of A coincides with $\{Q_A\} =$

$\{\chi_A\}$ for measurable $A \subseteq \mathbb{R}$. Finally, $\{\chi_A\}$ is isomorphic to the atomless measure algebra $\mathrm{Bor}(\mathbb{R})/\mathcal{N}$ (cf. Appendix A), where $\mathcal{N}$ is an ideal of Lebesgue measure null sets. Thus B_Q is atomless as well.

Remark 23. Observe that similar reasoning may be applied to the momentum operator P and its block B_P; moreover, it can be generalized to $L^2(\mathbb{R}^n)$ for any $n \geq 1$ by introducing B_Q as a block containing all position "coordinates" $Q_1, Q_2, \ldots, Q_n$. Similarly, the family $\{Q_i\}$ constitutes a maximally commuting system of observables on $L^2(\mathbb{R}^n)$ and the same conclusion holds.

We generalize above to the proposition, that every block B_Q containing the full family of position operators is isomorphic to some measure algebra.

Theorem 19. It holds that $B_Q \cong \mathrm{Bor}(\mathbb{R}^n)/\mathcal{N}$ for some $n \in \mathbb{N}$.

Proof. It is a direct consequence of Theorem 3 that B_Q is isomorphic to $\mathrm{Bor}(X)/\mathcal{N}$ for some measure space (X, μ). By Riesz theorem, $L^2(\mathbb{R}^n, \mathrm{d}^n x) \cong L^2(X, \mu)$ for separable, infinite-dimensional $L^2(X, \mu)$; since the isomorphism preserves blocks, the result follows. □

Nevertheless, one has to bear in mind that "block atomicity" is not an invariant associated with an operator. This is due to the fact that an operator may belong to several blocks simultaneously, as illustrated by the following

Example 8. Let $\mathcal{H} = L^2(\mathbb{R}^3)$ and $B_H \subseteq \mathcal{L}(\mathcal{H})$ be a block containing a free-particle Hamiltonian $H = P_1 + P_2 + P_3$. Then B_H is either atomless or contains at least one atom. To see that, observe that H by itself does not represent a maximally commuting system of observables. One obvious way of completing H to a block is to pick spectral resolutions for P_1, P_2, P_3; as they commute with H, we obtain an atomless block as a result. Surprisingly, for any operator A that does not form a maximally commuting system, it is always possible to completement A with a pure-point spectrum operator B, such that $\{A, B\}$ is already maximal [126]. Applying this to H we obtain a block $\{H, B\}$ with at least one atom, namely the one corresponding to one-dimensional projection in spectral resolution of B.

One can show that, in a way, atomic and atomless algebras exhaust all types of Boolean algebras of projections in the following sense [34]

Lemma 6. Every Boolean algebra B of projections can be decomposed as

$$B = B_a \otimes B_c,$$

where B_a is an atomic Boolean algebra (generated by finitely or infinitely many atoms) and B_c is an atomless Boolean algebra, isomorphic to measure algebra $\mathrm{Bor}([0,1])/\mathcal{N}$.

3.2 Automorphisms of $\mathcal{L}(\mathcal{H})$ and the Calkin algebra

In the end of this chapter, we gather several observations on the automorphisms of $\mathcal{L}(\mathcal{H})$, that might be of importance as soon as substructures of $\mathcal{L}(\mathcal{H})$ serve as parametrizations for a smooth spacetime manifold (see Chapter 5).

Let $\mathrm{Aut}(\mathcal{L}(\mathcal{H}))$ denote the group of automorphisms of $\mathcal{L}(\mathcal{H})$, where $\phi \in \mathrm{Aut}(\mathcal{L}(\mathcal{H}))$ whenever ϕ is bijective and preserves order and orthogonality, i.e.

$$M_1 \subseteq M_2 \implies \phi(M_1) \subseteq \phi(M_2) \text{ and } \phi(\neg p) = \neg\phi(p).$$

Define also Aut($\mathcal{B}(\mathcal{H})$) to be the group of automorphisms of $\mathcal{B}(\mathcal{H})$, where $\phi \in$ Aut($\mathcal{B}(\mathcal{H})$) if ϕ is a linear or antilinear bijection and

$$\phi(a^\dagger) = \phi(a)^\dagger \text{ and } \phi(ab) = \phi(a)\phi(b).$$

It can be shown that Aut($\mathcal{B}(\mathcal{H})$) is actually isomorphic to Aut($\mathcal{L}(\mathcal{H})$) if $\dim(\mathcal{H}) \geq 3$ (see [44] for a thorough discussion on this and the other automorphism groups related to Aut($\mathcal{L}(\mathcal{H})$)). Many properties of Aut($\mathcal{L}(\mathcal{H})$) can be deduced using the lattice version of Wigner's theorem [176]

Theorem 20. Every $\phi \in$ Aut($\mathcal{L}(\mathcal{H})$) is of the form

$$\phi_u(p) = upu^{-1}, \tag{3.2}$$

where $u \in \mathcal{B}(\mathcal{H})$ is unitary or antiunitary.

By Theorem 20 it is easy to see that every automorphism preserves commutativity, since $\phi([p,q]) = [\phi(p), \phi(q)]$. Furthermore, automorphisms preserve the completeness and maximality:

Proposition 1. Let $\phi \in$ Aut($\mathcal{L}(\mathcal{H})$) and $B \subseteq \mathcal{L}(\mathcal{H})$ be a complete (maximal) Boolean algebra. Then $\phi(B)$ is also a complete (maximal) Boolean algebra.

Proof. First, observe that $\phi(B)$ is a poset, as a subset of Aut($\mathcal{L}(\mathcal{H})$). As an automorphism, $\phi(B)$ contains all suprema and infima, hence $\phi(B)$ is a lattice. Observe $\phi(0) = 0 \in \phi(B)$ and $\phi(1) = 1 \in \phi(B)$ making $\phi(B)$ bounded. As ϕ preserves distributivity, $\phi(B)$ is a bounded, distributive lattice, thus a Boolean algebra.

Suppose B is complete and let $\{\phi(p_i)\}_{i\in I} \subseteq \phi(B)$. It is easy to see that $\phi(B)$ is complete as well, since [89, Lemma 1, pg. 143]

$$\phi\left(\bigwedge p_i\right) = \bigwedge \phi(p_i), \quad \phi\left(\bigvee p_i\right) = \bigvee \phi(p_i)$$

Finally, suppose that B is maximal (i.e. B is a block) and let $p \in \phi'(B)$. Then by Theorem 20 we have $\forall q \in \phi(B)\, \exists r \in B\, (q = uru^\dagger)$ and from $[p,q] = 0$ it holds $[p, uru^\dagger] = [u^\dagger pu, r] = 0$. Since B is a block, it holds $u^\dagger pu \in B$, therefore $\phi(u^\dagger pu) = p \in \phi(B)$ and $\phi(B)$ is also maximal, thus it is a block. □

In Chapters 4 and 5 we will argue that algebra homomorphisms between elements of BSub($\mathcal{L}(\mathcal{H})$) should correspond to smooth maps between open subsets of $\mathbb{R}^4$. This way, diffeomorphisms of $\mathbb{R}^4$ are to be represented by automorphisms of $\mathcal{L}(\mathcal{H})$ in this picture. Surprisingly, there is a sequence of interesting results relating Aut($\mathcal{L}(\mathcal{H})$) with a Cohen forcing.

Recall that $\mathcal{P}(\mathbb{N})/\text{Fin}$ is a Boolean algebra and it is easy to verify that it is atomless. Here, Fin is the Fréchet ideal of finite subsets of $\mathbb{N}$, thus $A, B \in \mathcal{P}(\mathbb{N})/\text{Fin}$ are equivalent if $A \Delta B$ is finite. Furthermore, let $\beta\mathbb{N}$ denote Stone-Čech compactification of $\mathbb{N}$ seen as a discrete topological space, i.e. $\beta\mathbb{N}$ is the set of all ultrafilters on $\mathbb{N}$, where every principal ultrafilter corresponds to some $n \in \mathbb{N}$. Consequently, a set $\beta\mathbb{N} \setminus \mathbb{N}$ is called *a remainder* and consists of non-principal (or free) ultrafilters on $\mathbb{N}$. At the same time, $\beta\mathbb{N} \setminus \mathbb{N}$ is a Stone space of $\mathcal{P}(\mathbb{N})/\text{Fin}$ (see Appendix A).

Let $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators given by

$$\mathcal{K}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid A[\mathcal{H}] \text{ is finite dimensional}\}.$$

The quotient $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called the *Calkin algebra*. It can be viewed as a noncommutative ("quantum") analogue for $\beta\mathbb{N}/\mathbb{N}$ [172] in the following sense. Suppose we fix an orthonormal basis for $\mathcal{H}$; then $\mathcal{H} = \ell^2(\mathbb{N})$. Let $\ell^\infty(\mathbb{N})$ be the set of bounded complex sequences; the elements of $\ell^\infty(\mathbb{N})$ can be naturally embedded into $\mathcal{B}(\ell^2(\mathbb{N}))$ as diagonal operators via $\ell^\infty(\mathbb{N}) \ni a \mapsto \hat{a} \in \mathcal{B}(\ell^2(\mathbb{N}))$ such that $\hat{a}(\psi) = (a_i\psi_i)$. Then, projections in the embedded $\ell^\infty(\mathbb{N})$ are projections on subspaces spanned by diagonal elements parametrized by subsets of $\mathbb{N}$, thus $\mathcal{L}(\ell^\infty(\mathbb{N})) \simeq \mathcal{P}(\mathbb{N})$. By Theorem 1, $\ell^\infty(\mathbb{N})$ as a commutative C^*-algebra is isomorphic to a concrete C^*-algebra $C(X)$ where X is locally compact Hausdorff and it can be shown that in our case $X = \beta\mathbb{N}$. Similarly, the set $c_0 = \ell^\infty(\mathbb{N}) \cap \mathcal{K}(\ell^2(\mathbb{N}))$ of complex sequences converging to 0 is a two-sided ideal of $\ell^\infty(\mathbb{N})$, so $\ell^\infty(\mathbb{N})/c_0$ is again a commutative C^*-algebra, and we have $\ell^\infty(\mathbb{N})/c_0 \simeq C(\beta\mathbb{N} \setminus \mathbb{N})$.

It is an interesting fact that many properties of Calkin algebra and related structures are of set-theoretic characteristics [172]. In particular, there are several results concerning automorphisms that depend e.g. on CH. One of the examples is that, assuming ZFC+CH, there is $2^{2^{\aleph_0}}$ many automorphisms of both $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ and $\mathcal{C}(\mathcal{H})$. The question whether all automorphisms of $\mathcal{C}(\mathcal{H})$ are *inner* (these are the maps of the form (3.2)) is independent of ZFC, and CH is sufficient to show the existence of an *outer* (i.e. not inner) automorphism. Similarly in the "classical" case, every bijection between cofinite sets $\mathbb{N} \setminus n \to \mathbb{N} \setminus m$, $n, m \in \mathbb{N}$ induces the so-called *trivial* automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. By analogy one shows that it is independent of ZFC whether all automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ arise this way, and again under CH one finds a nontrivial automorphism. Observe that via Stone duality (see Appendix A), automorphisms of $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ correspond to self-homeomorphisms of $\beta\mathbb{N}$ and $\beta\mathbb{N} \setminus \mathbb{N}$, respectively. In the Chapter 7 we provide an outlook for the possible future research on this topic in the context of exotic smoothness.

Chapter 4

Extending the universe

The introduction given in Chapter 1 raised questions about the conventional approach to quantum mechanics. We have previously argued that, aside from the remarkable alignment between theoretical and experimental physics across various domains (and perhaps a degree of simplicity), there is no inherent reason to refrain from available mathematical tools to address some gaps in our current comprehension of the natural world. In the current chapter, we elaborate on the possibility that a quantum system attributed to the assumed initial singularity can distinguish the large-scale structure of our observable Universe. Such a hypothesis gains support from the assumption that fundamental elements underpinning differential geometry, such as real numbers, are not absolute in the sense of ZFC models. This nonabsoluteness is a direct consequence of quantum mechanics' formalism, which naturally employs the tools we introduced in Chapters 2 and 3. As a byproduct, the approach gives an opportunity to shed new light on the problem of cosmological constant value, a topic we explore in this chapter; most of these results are contained in [108]. In what follows we will also use insights into the structure of $\mathcal{L}(\mathcal{H})$ and $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ given in Chapter 3. Results presented here are based mainly on [108][103].

4.1 From Boolean algebras of projections to forcing extensions

Currently, it is a widespread perspective within the physics community that our Universe originated from a singular state S of high energy density through the process called Big Bang, followed by rapid cosmological inflation phase. Given the extreme conditions assumed to prevail near the start of this process (e.g. distances smaller than the Planck length, energies larger than the Planck energy etc.), it is widely accepted that an accurate description of S should involve a foundational theory that combines quantum mechanics and general relativity in appropriate regimes, as both theories are believed to break down at S. While we do not have an access to such a fundamental theory, we make rather mild yet speculative assumptions about S:

- **A1** Physics at various scales can be generically expressed within distinct mathematical "universes" (e.g. topoi, ZFC models). In particular, the smoothness in **A2** may depend on the specific structure of real numbers inside the "universe" it is formulated in.
- **A2** Spacetime at high energies still can be modelled as a smooth manifold.
- **A3** Hilbert space approach in the sense of structures used (e.g. C^*-algebra of observables, orthomodular lattice of yes-no propositions) is likely to be present in a description of S.

Therefore, let us start with the quantum-mechanical system S, described by a Hilbert space $\mathcal{H}$. By **A3**, we demand the position Q and momentum P operators to be defined on

$\mathcal{H}$, hence by Remark 5 it follows that $\dim(\mathcal{H}) = \infty$. Also

$$\mathcal{H} \simeq L^2(\mathbb{R}^n) = \underbrace{L^2(\mathbb{R}) \otimes \ldots \otimes L^2(\mathbb{R})}_{n-\text{times}} \tag{4.1}$$

Note that at this point, we have refrained from making any assumptions about the number of dimensions n, which directly corresponds to spacetime dimensionality. Specifically, we have not asserted $n = 4$ thus far. Another important remark is that the real line $\mathbb{R}$ in (4.1) encompasses here both the quantum-mechanical parameter space and the coordinate space employed to model a spacetime as a smooth manifold. We will see that a distinction we have just established will give rise to nontrivial characterization of the spacetime manifold itself.

Let B_Q and B_P denote atomless blocks that contain P and Q, respectively, in the sense of their spectral resolutions (recall that this way we assume B_Q, B_P to include complete families of Q and P coordinates, respectively). It is evident that $P \notin B_Q$ and $Q \notin B_P$ since Q, P do not commute. Recall that due to Lemma 3, self-adjoint operators contained in B_Q and B_P are in one-to-one correspondence with the real numbers $\mathbb{R}_{B_Q}, \mathbb{R}_{B_P}$ from $\mathrm{Sh}(B_Q), \mathrm{Sh}(B_Q)$, respectively. We shall refer to $\mathbb{R}_{B_Q}, \mathbb{R}_{B_P}$ as *Boolean quantum real numbers*. By Example 7, both B_P and B_Q are atomless, hence they support non-trivial forcing (see Lemma 21).

Let us assume the existence of ultrafilters U_Q, U_P on B_Q, B_P over the universes V_Q, V_P, respectively. Then we obtain the following, two-valued models:

$$\mathrm{Sh}(B_Q)/U_Q = V_Q[U_Q], \quad \mathrm{Sh}(B_P)/U_P = V_P[U_P] \tag{4.2}$$

Consequently, the two-valued models $V_Q[U_Q]$, $V_P[U_P]$ contain the objects of real numbers $\mathbb{R}[U_Q], \mathbb{R}[U_P]$, which we call *quantum real numbers*.

Remark 24. As shown in Example 8, a block B_H containing the free-particle energy H may contain an atom; by Lemma 21, it leads to trivial forcing and therefore

$$\mathrm{Sh}(B_H)/U_H = V_H[U_H] = V_H, \quad \mathbb{R}[U_H] = \mathbb{R}_H \tag{4.3}$$

for every generic ultrafilter U_H on B_H over V_H.

Recall that both B_Q and B_P are measure algebras, isomorphic to atomless $B_n = \mathrm{Bor}\,(\mathbb{R}^n)\,/\mathcal{N}$ (see Theorem 19). By Lemma 21, B_n leads to nontrivial forcing. It follows that [19]

Lemma 7. For every nontrivial forcing extension of the real line $\mathbb{R}$ in V to $\mathbb{R}[U]$ in $V[U]$ for $U \in B_n$ generic over $\mathrm{Sh}\,(B_n) \cong V^{B_n}$, it holds that all measurable subsets of $\mathbb{R}$ in the extended model $V[U]$ have measure zero.

We consider the above lemma, together with Example 8 as a suggestive basis for the following line of reasoning: various physical quantities such as large-scale gravitational fields, or quantum fields, may propagate through spacetime parametrized by real numbers within distinct models, particularly in forcing extensions. Nevertheless, as shown in Example 8, certain specific forms of energy, corresponding to Hamiltonians contained in blocks possessing atoms, propagate in spacetime parametrized by real numbers from the trivial forcing extension (4.3).

We propose a cosmological model that offers an interpretation of the inflationary phase in the evolution of Universe from the model-theoretic perspective. This provides a alternative viewpoint regarding the cosmological constant problem. The subsequent section is dedicated to a more thorough analysis of this subject from the quantum-mechanical standpoint.

4.2 Vacuum energy vanishes

In Chapter 1 we briefly introduced how the cosmological constant (CC) problem confronts quantum mechanics, specifically quantum field theory, with classical physics, particularly in the realm of cosmology, as it appears to challenge the way quantum vacuum interacts via gravitation on a large scale. In the current section we leverage the conclusions drawn of Section 4.2 to suppress the substantial magnitude of zero-point fluctuations of quantum fields. By doing so, the significant discrepancy between quantum-mechanical and macroscopic perspectives on CC vanishes to some extent (however, the puzzle of a tiny, non-zero value of CC remains, and we will address this in Section 4.4). To achieve this goal, we first describe the problem of CC in a greater detail and elaborate on the fact it falls into the intersection of cosmology and quantum field theory.

Let us start with a short historical note. It is widely recognized that once Einstein had formulated his equations describing the interplay between spacetime's geometry (represented by Ricci and scalar curvature $R_{\mu\nu}$, R and the metric $g_{\mu\nu}$) and its matter-energy content (represented by the energy-momentum tensor $T_{\mu\nu}$):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4.4}$$

he was dissatisfied with the generic implication of (4.4), which indicated that the universe would either expand or contract. On the other hand, one has a freedom to introduce a specific term $\Lambda g_{\mu\nu}$ with $\Lambda = \text{const}$ in (4.4), what gives

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4.5}$$

What Einstein initially regarded as an appealing aspect of his equations was the possibility to force a universe to be static through (4.5). In fact, for an isotropic and homogeneous universe characterized by radius $a(t)$, one can achieve $\dot{a}(t) = 0$ by setting $\Lambda = 4\pi G\rho = a^{-2}$, with ρ representing energy density [27]. The drawback, however, was that such solution was inherently unstable [59, 127]. Several years later, the work of Hubble in 1929 decisively refuted static solutions and gave evidence for the expansion of Universe, as distant galaxies were observed to redshift [87]. All of these developments led Einstein to summarize his introduction of a particular $\Lambda \neq 0$ in (4.5) as his "greatest blunder". Ironically, it is worth noting that various measurements conducted to this day have indeed confirmed nonzero, small value of Λ, although differing from the one proposed by Einstein, obviously. So far, we have not encountered any significant issues with the experimental value of CC, primarily because we have regarded Λ as a free parameter of the theory. However, as previously discussed in Chapter 1, once we consider all potential sources for Λ, including quantum effects, it appears impossible to predict the actual value of Λ by means of any "canonical" method, and we explain what is meant by this in the following.

Let us discuss how quantum mechanics enters Einstein equations (4.5). Due to Lorentz invariance of the vacuum, we have [173]

$$\langle 0| T_{\mu\nu} |0\rangle = -\rho_{\text{vac}} g_{\mu\nu}, \tag{4.6}$$

where ρ_{vac} denotes a vacuum energy density; the equation (4.6) holds for all existing fields. It is reasonable to assume that the vacuum energy gravitates just like any other form of energy, in accordance with the equivalence principle. Thus, plugging (4.6) to (4.5), one obtains

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \underbrace{\left(\Lambda + 8\pi G\rho_{\text{vac}}\right)}_{\Lambda_{\text{eff}}} g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Observe that vacuum energy acts exactly as cosmological constant, therefore we may as well define

$$\rho_B = \frac{\Lambda}{8\pi G} \tag{4.7}$$

to be the *bare* vacuum density, corresponding to the *bare* cosmological constant parameter stemming from (4.5). Consequently, we introduce the effective cosmological constant $\Lambda_{\text{eff}} = \Lambda + 8\pi G\rho_{\text{vac}}$ and it is Λ_{eff} that can be directly observed. Recall that term Λ is merely a free parameter in (4.5), without any reference to quantum-mechanical description. At the same time, it is natural to represent $8\pi G\rho_{\text{vac}}$ by zero-point fluctuations of all quantum fields, hence quantum field theory enters cosmological regime here.

To estimate the contribution of quantum fields to Λ_{eff}, it is instructive to examine a simple case involving a real scalar field ϕ with mass m and a potential $V(\phi) = \frac{1}{2}m^2\phi^2$ [131]. The standard solution of Klein–Gordon equation of motion, given by the combination of creation and annihilation operators, leads to the vacuum energy density

$$\langle\rho_{\text{vac}}\rangle = \frac{1}{2}\int \mathrm{d}^3\mathbf{k}\frac{1}{(2\pi)^3}\omega(k) = \frac{1}{4\pi}\int_0^\infty \mathrm{d}k\, k^2\sqrt{k^2+m^2}, \tag{4.8}$$

where $\omega(k) = \sqrt{k^2+m^2}$ and $k = (k^0, \mathbf{k})$ is the four-momentum. Thus, plugging (4.8) into Λ_{eff} one arrives at

$$\Lambda_{\text{eff}} = \Lambda + \frac{8\pi G}{4\pi}\int_0^\infty \mathrm{d}k\, k^2\sqrt{k^2+m^2}, \tag{4.9}$$

which is manifestly infinite, due to UV-divergence of the integral (4.8). In fact, quantum field theory abounds in such divergent expressions and infinite vacuum energy on a flat Minkowski spacetime is not an exception. Observe that in case of Einstein equations (4.4), it causes the curvature of spacetime to be infinite. Typically, in the absence of gravity, it is argued that absolute vacuum energy is not observable and actually can be renormalized to any value. Indeed, phenomena like the Casimir effect or Lamb shift have been demonstrated to depend only on vacuum energy *differences*. However, the case of (4.19) is different, as the equivalence principle suggests that absolute value of vacuum energy matters here and it should therefore gravitate like any other form of energy. Nonetheless, UV-divergences such as (4.8) are usually recognized as a feature of applying given low-energy theory "all the way down", beyond its applicable range, and the entire framework of regularization and renormalization is devised to manage these infinities and extract meaningful, observable insights from the theory. To illustrate this, let us follow the most naive regularization approach and impose a hard momentum cut-off M on (4.8):

$$\langle\rho^M\rangle = \frac{1}{4\pi}\int_0^M \mathrm{d}k\, k^2\sqrt{k^2+m^2} = \frac{M^4}{16\pi^2}\left(1 + \frac{m^2}{M^2} + \ldots\right), \tag{4.10}$$

where we expanded the integral with respect to small parameter m^2/M^2. In other words, we assume that at energy scale M our current theory breaks down, and has to be replaced with a more fundamental one. Suppose that $M = M_P$, i.e. we treat the theory as valid and effective down to Planck's energy $M_P \sim 10^{18}$ GeV; this gives the prediction of QFT of roughly

$$\langle \rho^{M_P} \rangle \sim 10^{18}\text{GeV}^4 \sim 10^{72}\text{GeV}^4. \tag{4.11}$$

Surprisingly, it was already known to Lemaître that experimentally one finds

$$\langle \rho_{\text{eff}} \rangle = \frac{\Lambda_{\text{eff}} c^2}{4\pi G} \sim 10^{-47}\text{GeV}^4. \tag{4.12}$$

Comparing the numbers (4.11) and (4.12), it is indeed valid to state that the frequently cited disparity of $\log(\langle \rho^{M_P} \rangle \,/\, \langle \rho_{\text{eff}} \rangle) \approx 120$ orders of magnitude appears quite severe. At the same time, one needs to take it with some caution, as there are several misconceptions associated with this disagreement. First, the scale $M = M_P$ is a viable choice only if we assume QFT remains a valid theory up to Planck scale. However, it can be shown that even with a lower energy cutoff, such as $M = m_p$ (where m_p is the proton's mass), the vacuum energy is already 46 orders of magnitude too large [131]. Secondly, our discussion thus far has been limited to a simplified model of scalar field, and it is clear that the discrepancy between (4.11) and (4.12) should involve all the physical fields we are aware of, including the scalar Higgs field, fermion (including quarks and leptons) and gauge boson fields, which together constitute the Standard Model of particle physics. Calculating carefully above contributions, one arrives at [131]

$$\langle \rho_{\text{eff}} \rangle \approx -2 \times 10^8\,\text{GeV}^4 + \rho_B + \rho^{\text{EW}} + \rho^{\text{QCD}} + \ldots, \tag{4.13}$$

where ρ_B represents the bare density (4.7) and the ellipsis stand for other, currently unknown contributions. It is worth noting that, assuming supersymmetry, the contributions of bosonic and fermionic fields to the cosmological constant exactly cancel each other [131]. However, supersymmetry must be broken, as we have not observed supersymmetric partners of masses comparable to known particles. In this case, imposing a 100GeV cutoff we obtain a discrepancy of 52 orders of magnitude [122]. Lastly, it is important to recognize that CC problem is frequently framed as the incorrect prediction of the CC value arising from QFT, treated effectively at any scale. However, one has to bear in mind that the *effective* (i.e. measurable) CC is defined through

$$\Lambda_{\text{eff}} = \Lambda + \Lambda_{\text{QFT}}, \tag{4.14}$$

and the common misconception is the assumption that both $\Lambda = 0$ and $\Lambda_{\text{eff}} = \Lambda_{\text{QFT}}$ simultaneously hold. Note that there is no inherent reason for zero-point energies to precisely match the value of Λ_{eff}. Indeed, it might just be the case that Λ, Λ_{QFT} are adjusted so that $\Lambda_{\text{eff}} \sim 10^{-47}\text{GeV}^4$ as in (4.12). It does not resolve the problem, though: one still has to investigate what is the mechanism of this remarkably precise fine-tuning, as it corresponds to adjusting two, possibly unrelated numbers up to 55-120 decimal places (depending on the method adopted).

The approach outlined above can be considered the most natural and canonical means of generating a source of cosmological constant from the standpoint of quantum field theory and general relativity. However, the persistence of CC problem indicates that either we cannot resolve it without uniting these theories, or there are substantial flaws in our logic (both points could be true in principle). Recall that to solve the CC problem, one has to either suppress the gravitational interactions of quantum zero-modes completely (and leave the bare CC as a free, classical parameter), or provide the source for fine-tuning (4.14). In the remainder of this section, we will show that it is feasible to pursue the former option, applying the tools developed in Section 4.2.

To continue, let us briefly discuss statements **A1**, **A2**, **A3**. Firstly, **A1** points that it might not be reasonable to assume physics can be adequately formalized in a single model that

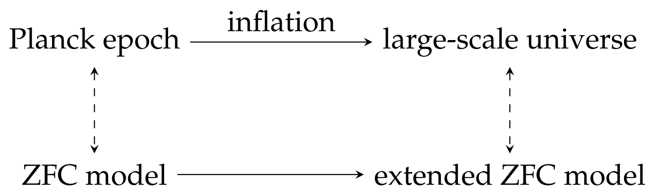


FIGURE 4.1: Model extensions representing the inflation

spans from cosmological scales all the way down to quantum-mechanical systems. Thus, we will introduce representations for these structures. Second, these counterparts will involve the relativization of certain parameters that are typically considered as constants, such as the object of real numbers. It is worth noting that, given an observable quantitity F parametrized implicitly by real line, every such modification has the potential to influence F, expressed as

$$F = \int d^3\mathbf{x} f(x) \equiv \int_R d^3\mathbf{x} f(x), \tag{4.15}$$

where R represents is a specific object associated with the real line. In principle, these objects are not identical when transitioning from one model to another, and consequently, the domain of integration has to be decided. Therefore, it is reasonable to reevaluate expressions like (4.15), whenever one suspects the parameter space changes. We argue that it also affects the notion of smoothness, which is the subject of **A2** and thus can be considered as a logical consequence of **A1**. Lastly, we have to agree on a starting point for building model extensions. According to **A3**, the logical structure of an initial, Planck-scale quantum system is that of an orthomodular lattice of projections. Therefore, the entire framework of Boolean-valued models described in Section becomes available, and we turn to detailed analysis in what follows.

Suppose that during inflation the change(s) of models occured, as illustrated on Fig. 4.1. Let the Planck-era spacetime be initially modeled by V. Following the inflation phase, we introduce extensions denoted as V_Q, V_H describing large-scale spacetime. Consequently, inflation entails a change of the real line $\mathbb{R}_V \mapsto \mathbb{R}_Q, \mathbb{R}_H$ as well. Recall that whenever Q and H give rise to atomless and atomic blocks, the forcing extensions V_Q, V_H are non-trivial and trivial, respectively. Furthermore, we have the following relationship

$$\mathbb{R}_V \subsetneq \mathbb{R}_Q, \quad \mathbb{R}_V = \mathbb{R}_H. \tag{4.16}$$

We see immediately that, whenever we refer to quantities parametrized by the pre-inflationary (non-extended) model, we should integrate over $\mathbb{R}_V$ rather than the extended $\mathbb{R}_Q$. This principle specifically applies to (4.8), resulting in the vacuum energy of a quantum scalar field as follows

$$\langle \rho_{\text{vac}} \rangle = \frac{1}{2} \int_{\mathbb{R}^3_V \subsetneq \mathbb{R}^3_Q} \frac{d^3\mathbf{k}}{(2\pi)^3} \omega(k), \tag{4.17}$$

and we integrate over $\mathbb{R}^3_V$ that is only a proper subset of large-scale $\mathbb{R}^3_Q$. To demonstrate that we no longer struggle with an UV-divergence here, we consider several facts. For the Lebesgue measure $\mu : \text{Bor}(X) \to [0,1]$ and $A \subseteq X$, the inner measure μ_* is defined by

$$\mu_*(A) = \sup\{\mu(U) \, : \, U \subseteq A, \, U \text{ — compact}\} \tag{4.18}$$

and the outer measure μ^* by

$$\mu^*(A) = \inf\{\mu(U) \,:\, A \subseteq U,\, U \text{ — open}\}.$$

(Note that A is measurable iff $\mu_*(A) = \mu^*(A)$ [19].) Recall that the set $A \subsetneq X$ is called meagre or Baire first category in X, whenever it is a countable union of nowhere dense subsets of X; we will need the following [114]

Lemma 8. In the case of random forcing, the real numbers $\mathbb{R}_V$ form a meagre subset of $\mathbb{R}_Q$.

Moreover, one shows [38]

Lemma 9. For the set $\mathbb{R}_V = 2^\omega \cap V$ of reals in the ground model one has $\mu_*(\mathbb{R}_V) = 0$, $\mu^*(\mathbb{R}_V) = 1$ and therefore $\mathbb{R}_V$ is nonmeasurable in V_Q.

Thus, Lemma 9 raises an issue, as it renders integrals over the entire $\mathbb{R}_V$ undefined, due to nonmeasurability of the domain. Consequently, one cannot compute integrals such as (4.17) in principle. On the other hand, recall that all the terms (4.13) producing cosmological constant should be considered as effective and are thus defined only on specific, bounded subsets of $\mathbb{R}_V$. This observation alone is sufficient to suppress all the contributions (4.13). To show this, we use the following

Lemma 10. For every compact, Lebesgue measurable subset S of the non-measurable set A with $\mu_*(A) = 0$ it follows $\mu(S) = 0$.

Proof. Immediate from definition (4.18). □

Corollary 2. By Lemmas 9 and 10, all compact measurable subsets of $\mathbb{R}_V$ have null measure, thus all integrals defined on compact $X \subset \mathbb{R}_V$ vanish.

As a consequence, quantum fields in the Planck epoch propagate "within" the model V, even though gravitational fields shall be parametrized by $\mathbb{R}[U_Q]$ (i.e. they propagate "within" the model $V_Q[U_Q]$). Now, the immediate implication of this postulate is that all entities parametrized by real numbers from the initial model will be reduced, as dictated by Lemma 9. It is important to recall that zero-energy modes of quantum fields have purely quantum-mechanical origin and, therefore, propagate in V. Conversely, both higher modes of quantum fields and large-scale gravity propagate "within" the extended model. As discussed earlier, in the standard approach, the zero-point energy without the cutoff (4.10) is UV-divergent. Nevertheless, due to the distinctive properties of the Lebesgue measure of subsets of real numbers defined inside the models, our approach yields

Corollary 3. Every contribution (4.13) with any finite UV-cutoff is suppressed, which gives immediately $\langle \rho_{\text{vac}} \rangle = 0$

Proof. By Lemma 9, all measurable subsets of $\mathbb{R}^3$ have null measure in $\mathbb{R}[U_Q]^3$. Moreover, $\mathbb{R}$ is a meagre subset of $\mathbb{R}[U]$. Thus, every integral of the form $\int\limits_{\mathbb{R}^3} (\cdot) \mathrm{d}^3x[U]$ calculated in the extended model vanishes, applying also to the particular case of $\int\limits_{\mathbb{R}^3 \subset \mathbb{R}[U]^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\sqrt{\mathbf{k}^2 + m^2}}{2}$. □

At first glance, this conclusion might appear as a drawback when compared to the currently accepted value (4.11). Indeed, even if we were to resolve the significant discrepancy (2.20), there is no possibility of making the contribution (4.10) assume any non-zero value due to Lemma 9. Therefore, in order to compute the realistic value of the cosmological constant, an alternative strategy needs to be developed. Recall that we previously defined $\Lambda_{\text{eff}} = \Lambda + 8\pi G\rho_{\text{vac}}$, suggesting the presence of two potentially separate contributions. Given our argument that the contribution from QFT vanishes, we deduce the following

Corollary 4. In the model-dependent approach, the effective cosmological constant is equal to the bare cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda = 8\pi G\rho_B \tag{4.19}$$

An obvious question emerges: can we provide a meaningful computation of (4.19) that is not only theoretically sound, but also consistent with experimental data? In the following section we demonstrate that this is indeed achievable and, surprisingly, one needs to take into account the smooth structure of spacetime that appears to stem from the lattice of subspaces of a Hilbert space of the initial quantum system.

4.3 Macroscopic smoothness from $\mathcal{L}(\mathcal{H})$

Recall that $\mathbb{R}$, defined as an Archimedean model for a Dedekind complete, ordered field, is unique up to an isomorphism. (Note that Dedekind completeness is a second-order property, unlike the first-order ZFC axioms, see Appendix B.) We claim that the distinction between real numbers within first-order ZFC models and the aforementioned second-order real line $\mathbb{R}$ might provide a formal analogy for the physical transition from real numbers that parametrize quantum-mechanical processes to the macroscopic real line that underpins large-scale physics.

Definition 2. We define the large-scale classical limits of quantum real numbers $\mathbb{R}[U_Q]$ and $\mathbb{R}[U_P]$ to be the substitutions $\mathbb{R}[U_Q] \to \mathbb{R}$ and $\mathbb{R}[U_P] \to \mathbb{R}$, respectively. In general, the large-scale classical limit of quantum real numbers $\mathbb{R}[U_A]$ from the model $\text{Sh}(B_A)/U_A$ for a self-adjoint A is the substitution $\mathbb{R}[U_A] \to \mathbb{R}$ and, similarly, the classical large-scale limit of $\mathbb{R}[U_A]^n$ is the substitution $\mathbb{R}[U_A]^n \to \mathbb{R}^n$, also denoted by $\mathbb{R}^n/\mathbb{R}[U_A]^n$.

Observe that in the case of a self-adjoint A contained in a block B_A with atoms, we obtain the substitution $\mathbb{R} \to \mathbb{R}$. This is the consequence of Lemma 21, giving that for any ultrafilter U_A on B_A over V we obtain

$$\text{Sh}(B_A) \cong V[U_A] \cong V.$$

Given an n-dimensional smooth manifold, we will provide a parametrization through blocks in $\mathcal{L}(\mathcal{H})$. Let $\text{BSub}(\mathcal{L}(\mathcal{H}))$ be the family of Boolean subalgebras $\mathcal{L}(\mathcal{H})$, and

$$\mathcal{U} = \{U_B \to \mathbb{R}^n : B \in \text{BSub}(\mathcal{L}(\mathcal{H}))\}$$

be a local regular open cover of an n-dimensional topological manifold $M^n_{\mathcal{L}(\mathcal{H})}$, and each U_B is the large-scale classical limit of some $\mathbb{R}^n_B$. We assume there exists the cover $\mathcal{U}$ with the following properties:

$$\forall U_B \in \mathcal{U}\, \exists B \in \text{BSub}(\mathcal{L}(\mathcal{H}))\ (U_B \to \mathbb{R}^n/\mathbb{R}^n_B) \tag{4.20}$$

$$\forall \mathcal{K} \text{ — open cover of } M^n_{\mathcal{L}(\mathcal{H})} \left(\mathcal{U} \subseteq \mathcal{K} \implies \bigcup \mathcal{U} \notin \mathcal{K}\right) \tag{4.21}$$

$$\exists B_1, B_2 \in \text{BSub}(\mathcal{L}(\mathcal{H}))\ (U_{B_1}, U_{B_2} \in \mathcal{U} \implies U_{B_1} \cup U_{B_2} \notin \mathcal{U}) \tag{4.22}$$

The condition (4.20) states that for every open $U_B \in \mathcal{U}$ there exists a Boolean algebra $B \in \text{BSub}(\mathcal{L}(\mathcal{H}))$ that provides an associated substitution. Secondly, (4.21) defines $\mathcal{U}$ to be unique, i.e. its union cannot be an element of a larger cover. The third point (4.22) above is built on the fact that in a nondistributive lattice $\mathcal{L}(\mathcal{H})$ there always exists at least one pair $\{Q, P\}$ of operators (e.g. consider the particle's position and momentum) giving rise to B_Q, B_P such that $B_Q \cup B_P$ is not a Boolean algebra anymore.

Definition 3. If $\mathcal{U}$ satisfies above conditions and is a smooth regular open cover of a smooth n-dimensional manifold $M^n_{\mathcal{L}(\mathcal{H})}$, then the pair $\left(M^n_{\mathcal{L}(\mathcal{H})}, \mathcal{U}\right)$ is called a *smooth manifold large-scale classical limit* (classical limit for short) of the lattice $\mathcal{L}(\mathcal{H})$.

While it is obvious that there could be several smooth manifolds M^n as candidates for a large-scale classical limit of given $\mathcal{L}(\mathcal{H})$, we will focus on the case of usually assumed $M^n = \mathbb{R}^n$. This brings us to an important result:

Theorem 21. If $\mathbb{R}^n$ is a smooth manifold large-scale classical limit of the lattice $\mathcal{L}(\mathcal{H})$ with $\dim(\mathcal{H}) > 2$, then it is an exotic R^4.

Proof. Let $\mathcal{U}$ be a regular topological open cover of a smooth $\mathbb{R}^4$. By (4.20), each $U_B \in \mathcal{L}$ is diffeomorphic to $\mathbb{R}^n$. By (4.22), there is a pair $B_1, B_2 \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ such that $U_{B_1} \cup U_{B_2} \notin \mathcal{U}$. Consequently, there exists a family $\mathcal{A} \subseteq \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ such that $\bigcup \mathcal{A} \notin \mathcal{U}$ and further $\bigcup \mathcal{U} \notin \mathcal{U}$. Thus $\bigcup \mathcal{U} \not\simeq \mathbb{R}^n$ as a smooth manifold, yet as a topological manifold it is homeomorphic to $\mathbb{R}^n$. Therefore $\bigcup \mathcal{U}$ has to be exotic smooth, but this is possible only in the case of $n = 4$. □

The following corollary justifies the term "large-scale classical limit".

Corollary 5. The lattice $\mathcal{L}(\mathcal{H})$ can be considered as a source of non-vanishing large-scale curvature of $\mathbb{R}^4$ and the curvature cannot be removed by any smooth coordinate transformation.

Proof. First, every exotic R^4 is a curved Riemannian manifold; if such R^4 had been flat, it would be diffeomorphic to $\mathbb{R}^4$ with a standard smooth structure, i.e. the atlas containing the global chart. Since R^4 is not diffeomorphic to the standard flat $\mathbb{R}^4$, there does not exist any smooth transformations $R^4 \to \mathbb{R}^4$. □

Recall that whenever an orthomodular lattice is distributive, it becomes a Boolean algebra and, in particular, it cannot model any non-trivial commutation relations. This is also reflected in the classical large-scale limit of $\mathcal{L}(\mathcal{H})$ as follows.

Theorem 22. If $\mathcal{L}(\mathcal{H})$ is a Boolean algebra, then the smooth structure of its large scale classical limit $\mathbb{R}^n$ is the standard $\mathbb{R}^n$.

Proof. Let $\mathcal{L}(\mathcal{H})$ be Boolean; then every Boolean subalgebra of $\mathcal{L}(\mathcal{H})$ can be extended uniquely to $\mathcal{L}(\mathcal{H})$, thus $\mathcal{L}(\mathcal{H})$ itself is a maximal Boolean algebra and since $\mathcal{U} = \{U_{\mathcal{L}(\mathcal{H})} \to \mathbb{R}^n\}$ is also a singleton, it holds that the conditions (4.21) and (4.22) are not satisfied and we have

$$\bigcup \mathcal{U} \cong U_{\mathcal{L}(\mathcal{H})} \in \mathcal{U}.$$

Hence $\mathbb{R}^n$ is not exotic, therefore it is standard smooth. □

To conclude, the (non-)Boolean structure of $\mathcal{L}(\mathcal{H})$ characterizes the spacetime smoothness given the conditions (4.20)-(4.22) and distinguishes whether the smooth structure is standard or not. Moreover, assuming spacetime topology to be $\mathbb{R}^n$ for any $n \in \mathbb{N}$, Theorem 21 immediately gives the four-dimensional spacetime.

4.4 Cosmological constant recovered

In the previous section we showed that the formalism of quantum mechanics deals inherently with structures, that could potentially interact with the smooth structure of a spacetime encompassing quantum systems. In the case of Euclidean $\mathbb{R}^4$, we established that the spacetime has to be exotic smooth. However, the approach does not identify specifically the smooth structure. Indeed, providing a more detailed connection between the lattice $\mathcal{L}(\mathcal{H})$ (or any other structure related to quantum mechanics) and spacetime smoothness calls for further work and we discuss this in Chapter 7. Instead, we explore here the implications of $M \cong \mathbb{R}^4$ being exotic smooth. It turns out that such a framework can naturally address the cosmological constant problem. We observe that cosmological constant can be linked with a topology change in such a manner that its value emerges as a topological invariant, and this theoretical result aligns remarkably well with experimental observations.

To start with, let us consider a toy model of a spacetime M with the topology $M \simeq S^3 \times \mathbb{R}$, which also serves as a cosmological model of the Friedmann–Lemaître–Robertson–Walker (FLRW) of isotropic and homogeneous universe. In this model, as we trace back in time, the sphere S^3 gradually contracts and converges eventually to the point-like region, which is to be identified as an initial singularity S. It is a reasonable assertion that a consistent depiction of this process must incorporate quantum-mechanical effects. As argued in the previous section, the singularity S (along with the surrounding region with a volume of at least ℓ_P^3) alters the large-scale smoothness, assuming the evolution of spacetime M originates from S. Accordingly we take M to be of the form $\Sigma \times \mathbb{R}$, although the spatial 3-dimensional submanifold Σ can dynamically change. A complete construction of this model is described in detail in [13]; here we provide a brief overview of the key points, that lead to agreement with recent experimental results from the PLANCK mission [150]. We follow several important observations gathered in [14], that constrain possible choices for spacetime. Firstly, we assume M to possibly undergo topological change(s) of the type $\Sigma_0 \times \mathbb{R} \to \Sigma_1 \times \mathbb{R}$, what leads to M being a 4-dimensional cobordism with $\partial M = \Sigma_0 \sqcup \Sigma_1$. Since there are mild hints toward finiteness of the spatial part of M [128], it seems reasonable to assume spatial Σ to be compact, 3-dimensional manifold without boundary. Furthermore, assuming compactness of Σ_0, Σ_1 at finite time, due to causality and the existence of Lorentz metric on M, one demonstrates that the boundary ∂M must have the same homology groups as S^3, i.e. both Σ_0, Σ_1 must be homology 3-spheres. Recall that the exotic smoothness in such a case is "localized" within an embedded Akbulut cork $A \subset M$ with a homology 3-sphere as a boundary (see Chapter 2). It can be shown [14] that a natural choice for an initial Σ_0 is then 3-dimensional sphere and the simplest choice for a homology sphere that is smoothly cobordant to S^3 is the Brieskorn sphere $\Sigma(2,5,7)$ defined by

$$\Sigma(2,5,7) = \{(x,y,z) \in \mathbb{C}^3 \, : \, x^2 + y^5 + z^7 = 0, |x|^2 + |y|^2 + |z|^2 = 1\}$$

and the 4-dimensional cobordism $W(S^3, \Sigma(2,5,7))$ is thus embedded in the Akbulut cork. Now, it is an essential part of [13] to recognize cosmological constant as a result of an embedding of a small exotic smooth $\mathbb{R}^4$ into $K3\#\mathbb{C}\bar{P}^2$ (i.e. the connected sum between $K3$ — unique 4-dimensional, compact, simply-connected manifold with a Ricci-flat metric, and $\mathbb{C}P^2$ being a complex projective plane). Namely, the embedding gives rise to the model that comprises a four-dimensional, compact cobordism M with 3-dimensional, spatial submanifold undergoing two topology changes:

$$S^3 \to \Sigma(2,5,7) \to P\#P \tag{4.23}$$

Considering the first topology change of (4.23), the time evolution $S^3 \times \{t_i\}$ for cross-sections $t_i \in \mathbb{R}$ is replaced with the wildly embedded sphere S^3 at the initial time t_0, evolving into the cross-section $\Sigma_{t_1} \subset S^3 \times_\Sigma \mathbb{R}$, where Σ_{t_1} is a homology 3-sphere. In the second topology change of (4.23), $P\#P$ stands for a connected sum of two copies of a Poincaré 3-homology sphere P (i.e. P has additionally a non-trivial fundamental group).

Interestingly, in [13] it is shown that the construction is canonical and each of the transitions (4.23) carries an inherent inflationary behaviour. Namely, if we assume a_0 to be the radius of an initial 3-sphere, one shows that expansions presented in (4.23) are specified by

$$a_0 \to a_0 \exp\left(\frac{3}{2\cdot \mathrm{CS}(\Sigma(2,5,7))}\right) \to a_0 \exp\left(\frac{3}{2\cdot \mathrm{CS}(\Sigma(2,5,7))} + \frac{3}{2\cdot \mathrm{CS}(P\#P)}\right), \qquad (4.24)$$

where $\mathrm{CS}(\cdot)$ is the Chern–Simons invariant of the hyperbolic 3-manifold (cf. [10, 13] for complete calculations). Here the particular invariants have values

$$\mathrm{CS}(\Sigma(2,5,7)) = \frac{9}{280}, \quad \mathrm{CS}(P\#P) = \frac{1}{60}.$$

To find the actual effects of inflation, one should make an assumption on the volume of initial 3-sphere. An educated guess dictates this size of the order of Planck's length ℓ_P, i.e.

$$a_0 = \sqrt{\frac{hG}{c^3}} = 1.61605 \cdot 10^{-35}\,\mathrm{m}. \qquad (4.25)$$

Thus by (4.24) and (4.25) we obtain the characteristic length of the first topological transition

$$a_1 = \ell_P \cdot \exp\left(\frac{3}{2\cdot \mathrm{CS}(\Sigma(2,5,7))}\right) = \ell_P \cdot \exp\left(\frac{140}{3}\right) = 3.0 \cdot 10^{-15}\,\mathrm{m}.$$

Observe that this should be interpreted as a spatial size of the universe, represented at this stage by $\Sigma(2,5,7))$, after first topological transition. The associated energy scale of the first transition is therefore

$$E_1 = \frac{\mathrm{hc}}{a_1} = 415\,\mathrm{MeV}.$$

It is interesting to observe that E_1 is of the order of a QCD energy scale $\Lambda_{QCD} \approx 200\mathrm{MeV}$ [163].

Crucially, through the embedding of small exotic $\mathbb{R}^4$, the approach [13] demonstrates that cosmological constant can be calculated purely topologically, and is related directly to the spatial curvature $\mathrm{curv}(M)$ of a 4-manifold M. In particular, let $W(\Sigma_1, \Sigma_2)$ be a cobordism between 3-manifolds, embedded in a hyperbolic 4-manifold. Then, a *cosmological constant of the cobordism* is defined through

$$\mathrm{curv}(\Sigma_2) = \mathrm{curv}(\Sigma_1) \cdot \exp(-2\theta),$$

where θ is the parameter dependent on the Chern–Simons invariant of Σ_2. In the case of (4.23), we have the cobordism encompassing two transitions between 3-manifolds, giving the (normalized) curvature of $P\#P$

$$\mathrm{curv}(P\#P) = \frac{1}{8\pi^2}\mathrm{curv}(S^3)\exp\left(-\frac{3}{\mathrm{CS}(\Sigma(2,5,7))} - \frac{3}{\mathrm{CS}(P\#P)}\right)$$

Accordingly, the cosmological constant is calculated in this approach as

$$\Lambda = \mathrm{curv}(P\#P) = \frac{1}{8\pi\ell_P^2}\exp\left(-\frac{3}{\mathrm{CS}(\Sigma(2,5,7))} - \frac{3}{\mathrm{CS}(P\#P)}\right) \approx 9.5\cdot 10^{-52}\,\mathrm{m}^{-2}$$

or, in Planck units, $\Lambda\cdot\ell_P^2 \approx 2.5\cdot 10^{-121}$. To confront the obtained cosmological parameter with experimental data from [150], we calculate the dark energy density parameter Ω_Λ:

$$\Omega_\Lambda = \frac{\Lambda}{\Lambda_{\mathrm{crit}}} = \frac{\Lambda c^2}{3H^2},$$

where H is the Hubble's constant and Λ_{crit} is the critical value that would make the spatial geometry flat [162]. Given the latest data on Hubble's constant $H = (67.36 \pm 0.54)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [150] we have

$$\Omega_\Lambda = \frac{c^5}{24\pi^2 hGH^2}\exp\left(-\frac{3}{\mathrm{CS}(\Sigma(2,5,7))} - \frac{3}{\mathrm{CS}(P\#P)}\right) \approx 0.9053$$

which is quite far from the experimental value [150]

$$\Omega_\Lambda^{\mathrm{exp}} = 0.6847 \pm 0.0073. \tag{4.26}$$

However, it can be demonstrated that the quantum corrections to the initial transition $S^3 \to \Sigma(2,5,7)$ are derivable from the Euler characteristic $\chi(A)$ of the Akbulut cork A with a boundary $\partial A = \Sigma(2,5,7)$. As A is contractible, we have $\chi(A) = 1$ and the correction is of the form

$$\exp\left(-\frac{\chi(A)}{4}\right) = \exp\left(-\frac{1}{4}\right). \tag{4.27}$$

Finally taking (4.27) into account we obtain

$$\Omega_\Lambda = 0.9053\cdot\exp\left(-\frac{1}{4}\right) \approx 0.7050$$

which is much closer to the value (4.26).

Remark 25. It is worth noting that the above approach also settles several other cosmological parameters, such as the α-parameter in the Starobinsky model, the scalar/tensor ratio and, more recently, bounds on neutrino masses [8, 14].

Chapter 5

Going categorical into exotic smoothness

In Chapter 3 we discussed the difference between quantum and classical physics from the perspective of logical structures. In particular, we demonstrated how one approximates the lattice $\mathcal{L}(\mathcal{H})$ with the family $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ of its Boolean subalgebras. In Chapter 4 we explained how the parametrization of spacetime M by $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ results in the exotic smoothness of M. As this approach may appear somewhat ambiguous, in this section we illustrate how the categorical approach might help in translating the structure of Boolean subalgebras of $\mathcal{L}(\mathcal{H})$ into a smooth atlas covering spacetime. It is a general rule that as soon as appropriate categories are found, some relations between seemingly distant notions might become functorial, giving rise to a better understanding. This aligns with the perspective presented in [30][29]. Additionally, we extend this discussion by exploring how distribution theory, formulated separately within **Set** and the Basel topos $\mathcal{B}$ may provide further insights into exotic smoothness, as worked out in [11].

5.1 The category pBool

We start with a categorical framework for OMLs, which, when addressed directly, appears to be slightly too restrictive and thus requires to be extended. Let **OML** be a category with orthomodular lattices as objects and lattice homomorphisms as arrows. As such, **OML** contains $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ as a poset subcategory, since every Boolean algebra is a distributive orthomodular lattice, cf. Appendix A. Our interest lies in understanding how $\mathcal{L}(\mathcal{H}) \in \mathrm{Ob}(\mathbf{OML})$ can be derived from $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ using the language of category theory. Intuitively, this derivation can be accomplished by adopting the concept of a pasting [72]. In category theory, the general method of constructing an object by combining smaller ones is known as a *coproduct* or a colimit in general. Let

$$\mathbf{D} = \{B \hookrightarrow \mathcal{L} : B \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))\}$$

be a diagram of inclusions of Boolean subalgebras into $\mathcal{L}(\mathcal{H})$ (see Fig 5.1). This makes $\mathbf{D}$ a cocone inside **OML**. Note that the collection of all cocones over $\mathbf{D}$ forms a distinct category and a limit for $\mathbf{D}$ is a terminal object in that category.

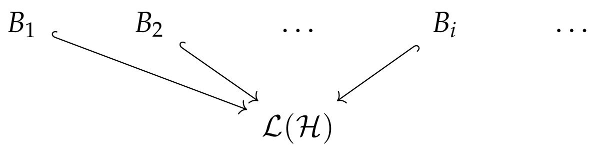


FIGURE 5.1: Pasting of Boolean subalgebras

Remark 26. The **OML** category is not (co)complete; in particular, tensor products do not exist there in general, thus it does not neccessarily hold that $L_1 \otimes L_2 \in \mathrm{Ob}(\mathbf{OML})$ for $L_1, L_2 \in \mathrm{Ob}(\mathbf{OML})$.

Fortunately, there exists a notion of a partial Boolean algebra, that generalizes the notion of OML (see Chapter 2). Recall that a partial Boolean algebra is essentially a set A with a Boolean algebra structure defined only "locally": it includes a reflexive and symmetric *compatibility* (commeasurability) relation $\odot \subseteq A \times A$ together with $0, 1$ and usual logical operations $\neg, \wedge, \vee$. "Locality" means that whenever a collection of elements are commeasurable, it is contained in some $T \subseteq A$ with a Boolean algebra structure imposed on T. The category **PBool** comes with partial Boolean algebras as objects and functions preserving commeasurability and algebraic structure as morphisms. Observe that totality of $\odot \subseteq A \times A$ makes A a Boolean algebra, similarly to the commutativity of an OML (actually, **OML** is a subcategory of **PBool**). Furthermore, **PBool** is (co)complete. Recall that due to Theorem 17, every OML is uniquely (up to isomorphism) characterized by the poset of its Boolean subalgebras. Since colimits are necessarily unique up to isomorphism in any category, one obtains the category-theoretic version of Theorem 17 [26]

Theorem 23. Every partial (complete) Boolean algebra is a colimit of its Boolean subalgebras.

Remark 27. Theorem 23 singles out logical structures (OML's, partial Boolean algebras). In contrary, there is no analogue in the case of algebraic structures such as C^*-algebras and von Neumann algebras; these can be determined by their commutative subalgebras only up to a commutator, that defines a so-called Jordan structure [52, 125].

5.2 The category nMfd

Let us discuss now the part of Section 2.3 that refers to spacetime smoothness. Again, we will rephrase crucial notions in category-theoretic terms. First of all, the notion of a n-dimensional smooth manifold X, i.e. n-dimensional topological manifold X with an atlas of charts $\phi_i : U_i \to \mathbb{R}^n$ for which transition maps $\phi_i \circ \phi_j^{-1}$ are smooth, can be characterized as follows: given an atlas $\{\phi_i : X \supseteq U_i \to V_i \subseteq \mathbb{R}^n\}$ it holds that

$$X = \bigsqcup V_i / \sim, \tag{5.1}$$

where $\phi_i(p) \sim \phi_j(p)$ whenever $p \in U_i \cap U_j$. Let us now illustrate how a manifold can be presented in two, dual pictures by the example [119].

Example 9. Let $X = S^2$; such a manifold may be viewed in a twofold way (see Fig. 5.2): S^2 is captured either by a commutativity of the following equalizer diagram

$$S^2 \hookrightarrow \mathbb{R}^3 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{R}$$

where $s(x, y, z) = x^2 + y^2 + z^2$ and $t(x, y, z) = 1$, which by commutativity gives $x^2 + y^2 + z^2 = 1$ and defines the sphere as a limit, or S^2 can be parametrized by the coequalizer diagram

$$S^1 \times (0, 1) \rightrightarrows D \sqcup D \longrightarrow S^2$$

where D is a two-dimensional disk and the commutativity of inclusion of a cylinder $S^1 \times (0, 1)$ is responsible for the quotient by the glued part of $D \cup D$, characterizing sphere by a colimit.

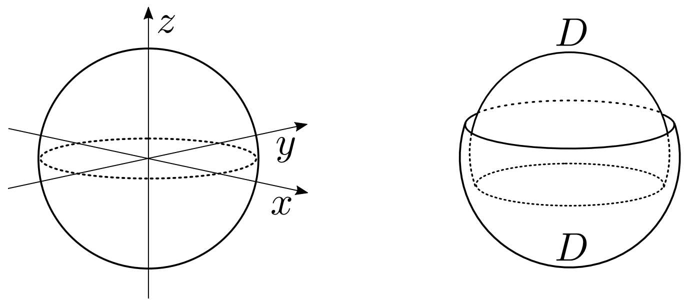


FIGURE 5.2: The limit (left) and colimit (right) representations of a sphere S^2

Observe that the second approach in the above is coordinate-free and emphasizes the role of "subobjects" of X, thus is closer to (5.1). Now, we generalize that to the form previously met in the case of orthomodular lattices. Let **nMfd** be a category of n-dimensional smooth manifolds and smooth maps between them. Then we have [120]

Theorem 24. Let $\mathbf{U_n}$ be a subcategory of **nMfd**, consisting of all open subsets of $\mathbb{R}^n$ and smooth maps between them. Then every object in **nMfd** is a pasting of objects from $\mathbf{U_n}$.

Categorically, there is a straightforward version of the above fact, generalizing the coequalizer diagram from Example 9 [115]:

Theorem 25. Any object M in **nMfd** is a coequalizer

$$\bigsqcup W_{ij} \rightrightarrows \bigsqcup W_i \longrightarrow M$$

where $W_i = f_i(V_i)$ and $W_{ij} = f_i(V_i \cap V_j)$.

Hence, by above theorem we finally obtain

Corollary 6. Every smooth manifold is a colimit of its atlas.

5.3 Functoriality of pBool → nMfd

We recognize Theorem 23 and Corollary 6 to provide us an opportunity to expand upon the findings in Chapter 4 on the exotic smoothness of the spacetime M. This goes beyond just having an atlas parametrized by Boolean contexts of $\mathcal{L}(\mathcal{H})$; the parametrization aligns categorically. This leads us to ask whether the colimit correspondence might be of functorial character. In other words, to what extent can we assign open subsets of $\mathbb{R}^n$ (i.e. objects in $\mathbf{U_n}$ hence in **nMfd**) to Boolean frames of reference (i.e. objects in $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ hence in **pBool**), such that the assignment follows the rules of a (covariant) functor $F : \mathbf{pBool} \to \mathbf{nMfd}$. Namely, we have

$$F(B) = U_B \in \mathrm{Ob}(\mathbf{U_n}) \subseteq \mathrm{Ob}(\mathbf{nMfd}) \quad \text{and} \quad F(f) \in \mathrm{Arr}(\mathbf{U_n}) \subseteq \mathrm{Arr}(\mathbf{nMfd})$$

for any $B \in \mathrm{Ob}(\mathbf{pBool})$ and $f \in \mathrm{Arr}(\mathbf{pBool})$, together with $F(f \circ g) = F(f) \circ F(g)$. (Note that by an analogy, the group $\mathrm{Aut}(\mathcal{L}(\mathcal{H}))$ would correspond to the group $\mathrm{Diff}(M)$, what we mentioned already in Chapter 3.) We point out that the existence of such a functor F is an open issue left for future research (see Chapter 7). Nevertheless we indicate two points that may be important in the construction thereof.

First, let us recall that $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ is not only a mere posetal category; it constitutes also a meet-semilattice, as shown in [78]. In other words, for any $B_1, B_2 \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ we have

$$B_1 \wedge B_2 := B_1 \cap B_2 \in \mathrm{BSub}(\mathcal{L}(\mathcal{H})).$$

In category-theoretic terms, that is to say that the pullback always exists, i.e. the square

$$\begin{array}{ccc} B_1 \cap B_2 & \hookrightarrow & B_2 \\ \downarrow & & \downarrow \\ B_1 & \hookrightarrow & \mathcal{L}(\mathcal{H}) \end{array}$$

commutes. For an analogous result in the context of smooth manifolds, we have to go to a subcategory **para−nMfd** of *paracompact* manifolds.

Remark 28. Recall that a topological space X is called *paracompact* if X is Hausdorff and every open cover of X has a locally finite open refinement (i.e. for every such cover $\{U_i\}_{i\in I}$ one finds a family $\{\tilde{U}_j\}_{j\in J}$ such that for every $x \in X$ there exists an open neighbourhood that intersects only finite subset of $\{\tilde{U}_j\}_{j\in J}$, and for every $i \in I$ there exists $j \in J$ such that $\tilde{U}_j \subseteq U_i$ [61].) In fact, this is the usual category to talk about smooth manifolds, in particular spacetimes, in the standard approach. This is due to the fact, that all second countable *and* Hausdorff topological spaces are necessarily paracompact [115].

Then, one can show that each object in **para−nMfd** admits a good open cover, i.e. given any $X \in \mathrm{Ob}(\mathbf{para{-}nMfd})$, all its chart intersections are diffeomorphic to $\mathbb{R}^n$. Therefore, under the functor $F|_{\mathbf{para-nMfd}}$ an intersection of elements of $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ should correspond again to some $U \in \mathrm{Ob}(\mathbf{U_n})$.

Second, let us consider the construction of adjoint functors [179]

$$L : \mathbf{Set}^{[\mathrm{Bor}(\mathbb{R})/\mathrm{BSub}(\mathcal{L}(\mathcal{H}))]^{op}} \underset{\longrightarrow}{\longleftarrow} [\mathrm{Bor}(\mathbb{R})/\mathcal{L}(\mathcal{H})] : R$$

where $[\mathrm{Bor}(\mathbb{R})/\mathcal{L}(\mathcal{H})]$ is a "quantum" comma category over $\mathcal{L}(\mathcal{H})$. This gives an opportunity of studying $[\mathrm{Bor}(\mathbb{R})/\mathcal{L}(\mathcal{H})]$ in terms of presheaves on a "classical" $[\mathrm{Bor}(\mathbb{R})/\mathrm{BSub}(\mathcal{L}(\mathcal{H}))]$ and vice versa. As the objects of $[\mathrm{Bor}(\mathbb{R})/L]$ are obviously the arrows of **pBool**, it seems reasonable to look for analogous adjoint in the category n-**Mfd**.

5.4 Exotic smoothness through Basel topos

To conclude this chapter, we discuss yet another categorical approach to exotic smoothness. In this context, we locally modify the spacetime structure using the Basel topos $\mathcal{B}$ we introduced in Chapter 2. Recall that the internal logic of this topos is intuitionistic, and it encompasses both invertible and nilpotent infinitesimal real numbers within the smooth real numbers object. One notable consequence is the unique way distribution theory is handled internally: all distributions become regular, effectively allowing us to treat distributions as if they were functions within $\mathcal{B}$.

Remark 29. The application of distributions interpreted as "regular" functions in physics is not a novel concept and has been previously proposed and explored using *non-standard analysis*, which we briefly discussed in Chapter 2. This approach has primarily been applied to quantum field theory, typically plagued with infinities, cf. [97, 140, 73, 177, 22].

Let us now give a brief overview of the work presented in [11], which established a connection between the previous considerations and the smooth structures of a manifold

$M \in \mathbf{Mfd}$ through embedding $s : \mathbf{Mfd} \to \mathcal{B}$. We will provide a broad outline of the key ideas; for detailed proofs cf. [11]. Let $U = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a smooth atlas for a smooth manifold M. To include the local modification, for each $\alpha \in I$ we impose either a map $s : \mathbf{Mfd} \to \mathcal{B}$ with $s : U_\alpha \mapsto s(U_\alpha)$ or an identity diffeomorphism $i : U_\alpha \to U_\alpha$ in **Set**. Let $A \subset I$ be a set of elements that index identity maps, so that we assign

$$U_\alpha \mapsto \begin{cases} i(U_\alpha) \simeq U_\alpha, & \alpha \in A \\ s(U_\alpha) \in \mathcal{B}, & \alpha \in I \setminus A \end{cases} \tag{5.2}$$

We say a map (5.2) is a $\mathcal{B}$-*cover* of a smooth manifold M if both $A \neq I$ and $A \neq \varnothing$, i.e. it is not composed *entirely* of maps of a single type. Further, a $\mathcal{B}$-*local smooth manifold* is a smooth manifold M such that every atlas of M has an underlying $\mathcal{B}$-cover $\mathcal{U}^{\mathcal{B}}$, in the sense that for every smooth cover $(M, \{U_{\alpha_i}\}_i)$ of M there exists a $\mathcal{B}$-cover $\mathcal{U}^{\mathcal{B}}$ such that

$$\forall U_\alpha \,(U_\alpha \simeq i(U_\alpha) \text{ or } U_\alpha \simeq \Gamma U_\alpha^{\mathcal{B}})$$

where $\Gamma : \mathcal{B} \to \mathbf{Set}$ is a global section functor (see Chapter 2). It is interesting to explore whether $\mathcal{B}$-local covers determine smoothness in any way. In the case of Euclidean $\mathbb{R}^n$, we have already established through the proof of Theorem 21 that one-element atlases characterize standard $\mathbb{R}^n$, thus we have

Proposition 2. If $M = \mathbb{R}^n$ and $\hat{M}$ is smooth, then $\hat{M} \simeq R^4$, where R^4 is a certain exotic smooth $\mathbb{R}^4$.

Remark 30. It follows from Proposition 2 that a $\mathcal{B}$-local smooth $\mathbb{R}^4$ must be exotic smooth. This raises an interesting question: can some (all) exotic smooth $\mathbb{R}^4$s be associated with such $\mathcal{B}$-local smooth $\mathbb{R}^4$s? This provides a promising starting point for developing tools to distinguish between exotic smooth structures on $\mathbb{R}^4$.

To relate this to distributions handled inside $\mathcal{B}$, let us consider the following simple observation. First, given a standard $\mathbb{R}^4$ and exotic R^4, all the smooth functions are always continuous regardless which smooth structure they refer to, i.e.

$$C^\infty(\mathbb{R}^4) \subset C^0(\mathbb{R}^4) \supset C^\infty(R^4).$$

Second, we have the following fundamental [141]

Lemma 11. $M \simeq N$ as smooth manifolds precisely when $C^\infty(M) \simeq C^\infty(N)$ as algebras.

For a particular case of M, N being an exotic smooth R^4 and a standard $\mathbb{R}^4$ respectively, we obtain a way to characterize exotic smoothness through the behaviour of smooth maps. Specifically, there exists a function $f \in C^0(\mathbb{R}^4)$ that is smooth with respect to exotic smoothness on $\mathbb{R}^4$ but only continuous with respect to the standard smoothness (and vice versa). The apparent asymmetry in the possibility of differentiation with respect to both smooth structures can be resolved with the help of distribution theory, which provides the tools to differentiate non-smooth functions through the integral representation (2.20). This observation is also related to Remark 30, as it helps to identify functions that lose their smoothness when transitioning between $\mathbb{R}^4$ and any R^4 (and vice versa), thereby distinguishing exotic R^4 structures from the standard one.

Let us now focus on interpreting merely continuous functions inside $\mathcal{B}$. Recall that given a $\mathcal{B}$-local atlas, at least one patch U_α is translated into $\mathcal{B}$ and therefore some transition functions $\phi_{\alpha\beta} : U_{\alpha\beta} \to U_{\alpha\beta}$ are to be replaced by $\phi_{\alpha\beta}^{\to\mathcal{B}} : U_{\alpha\beta} \to U_{\alpha\beta}^{\mathcal{B}}$. To formulate the transition to $\mathcal{B}$, we have at our disposal both geometric morphism (Γ, Δ) and the full and faithful embedding map $s : \mathbf{Mfd} \to \mathcal{B}$, cf. (2.18). We say a $\mathcal{B}$-local smooth structure $U^{\mathcal{B}}$ on

$\mathbb{R}^4$ is *smoothly equivalent* to a smooth structure U on $\mathbb{R}^4$ if every function that is $U^{\mathcal{B}}$-smooth is also U-smooth. We say $U^{\mathcal{B}}$ is **Set**-*invariant* if there exists smooth structure V on $\mathbb{R}^4$ such that $U^{\mathcal{B}}$ is smoothly equivalent to V. With above definitions, the local modification of a smooth structure on $\mathbb{R}^4$ leads to the following

Theorem 26. For every $\mathcal{B}$-local smooth cover $U^{\mathcal{B}}$ of $\mathbb{R}^4$ that is **Set**-invariant, there exists a continuous function $f \in C^0(\mathbb{R}^4)$ that is $U^{\mathcal{B}}$-smooth. Furthermore, if the **Set**-invariant structure $U^{\mathcal{B}}$ is smoothly equivalent to some V, then V is not diffeomorphic to U.

Corollary 7. Theorem 26 is one possible way to construct exotic smooth structure on $\mathbb{R}^4$ by considering continuous functions, although smooth structures that arise this way are not known to be pairwise exotic.

As above results involve transitioning (and invariance) between topoi **Set** and **B**, it is interesting to comment here on the *principle of general tovariance*, proposed by Landsman in [80]. As the name suggests, to-variance refers to general co-variance, introduced by Einstein, which we mentioned in Chapter 1. While the latter means that laws of physics should be preserved under smooth coordinate transformations, the former states that the same rule should apply to appropriate transformations between topoi. To identify what type of transformations could be termed "appropriate", recall that to ensure frame homomorphisms preserve Heyting algebra structure, only so-called *geometric* theories are allowed. As topoi generalize frames, by an analogy we say a theory is *geometric* if it can be axiomatized with sentences of the form $\forall x\,(\phi(x) \to \psi(x))$, where ϕ, ψ are geometric formulae (again, these are formulae in the language of intuitionistic propositional logic with finite meets) (see Appendix B). Consequently, a geometric morphism is a pair (g_*, g^*) of functors $g_* : \mathcal{T}_1 \to \mathcal{T}_2$, $g^* : \mathcal{T}_2 \to \mathcal{T}_1$ that generalizes a frame map in that g^* preserves finite limits. As a consequence, g^* preserves also any geometric theory. Finally, the principle of general tovariance states that laws of physics must be definable in any topos $\mathcal{T}$ with NNO, and these laws must be preserved with respect to geometric morphisms. Thus, given such morphism $(g_*, g^*) : \mathcal{T} \to \mathcal{T}'$, one cannot empirically distinguish between $\mathcal{T}$ and $\mathcal{T}'$. This applies also to the Bohrification program (with minor details concerning nongeometricity of the completion of a C^*-algebras). In particular we have

Lemma 12. Let $g : \mathcal{T}_1 \to \mathcal{T}_2$ be a geometric morphism. If $\mathbb{N}_2$ is a natural number object in $\mathcal{T}_2$, then $\mathbb{N}_1 = g^*\mathbb{N}_2$ is a natural number object in $\mathcal{T}_1$.

Since the pair (2.18) gives rise to a geometric morphism $g = \Delta \dashv \Gamma$ between $\mathcal{B}$ and **Set** and $g^*\mathbb{N}_{\mathbf{Set}} = \mathbb{N}_{\mathcal{B}}$, by the fact that N in $\mathcal{B}$ is not isomorphic to $\mathbb{N}_{\mathcal{B}}$ we have

Corollary 8. The object N of smooth natural numbers in $\mathcal{B}$ is not preserved under geometric morphism $\Delta \dashv \Gamma$. Moreover, given any topos $\mathcal{T}$ with a natural number object $\mathbb{N}_{\mathcal{T}}$, a geometric morphism $g : \mathcal{B} \to \mathcal{T}$ necessarily does not preserve N.

As discussed in [11], the above fact is particularly interesting in the context of tovariance. Namely, recall that e.g. distribution theory in $\mathcal{B}$ as described in Chapter 2 makes heavy use of the smooth natural numbers N. More importantly, recall that currently it is not known which exotic R^4 correspond to certain **Set**-invariant $\mathcal{B}$-local smooth structures. Thus, a conjecture would be as follows: *If experimental evidence points at exotic smooth R^4 described in Chapter 4 and this R^4 is equivalent to certain* ***Set****-invariant $\mathcal{B}$-local smooth structure, then clearly a tovariance principle must be broken.*

Finally, note there is a possible connection between exotic smooth structures that a $\mathcal{B}$-local modification generates and singularities that emerge typically in physics due to the problem of multiplying distributions with intersecting supports [30]. The key insight here is that, thanks to the interpretation of distribution theory within $\mathcal{B}$, every UV-divergence involving the multiplication of distributions in **Set** can theoretically be translated and resolved in $\mathcal{B}$. We provide some further remarks on this topic in Chapter 7.

Chapter 6

Into the algorithmically random world(s) and beyond

6.1 Preliminaries

In this chapter we demonstrate how the method presented in preceding sections offers new insights into the issue of randomness within the realm of quantum mechanics, a topic we briefly discussed in Chapters 1 and 2. The current chapter is based on the work [111]. To begin, we will provide some broad observations that concern both philosophical and technological aspects; for a thorough discussion see [25], [118] or [101]. The concept of randomness in the natural world is a fascinating subject on its own, as one distinguishes between processes that initially appear random and those that are believed to be irreducibly random. The former comprise a vast group of classical stochastic and chaotic systems. Indeed, it is widely recognized that classical systems, such as a container filled with particles, or even a relatively simple system like a double pendulum, are described by differential equations of motion. In principle, given specific boundary conditions, the Picard–Lindelöf theorem states that the system's future behaviour is entirely predictable, provided we possess a perfect knowledge about the system's initial state (with a caveat of a Lipschitz condition, see Remark 32). It is worth noting that over the years, there has been extensive exploration in various contexts regarding the extent to which full knowledge is actually necessary for making meaningful predictions, with thermodynamics as a good example.

Remark 31. The term "initial conditions" in the realm of classical physics must be handled cautiously, as the perfect knowledge of positions and momenta is fundamentally constrained by quantum effects (most importantly Heisenberg's uncertainty principle). Setting aside these considerations, obtaining complete information on the initial conditions would necessitate knowledge on the positions and momenta of every particle in the observable Universe once we take gravitational interactions into account.

Remark 32. In fact, Picard–Lindelöf theorem is not sufficient to guarantee complete determinism for all classical systems governed by Newtonian mechanics. One of the exceptions is the case of a particle being shot off to infinity in a finite time (often referred to as a *a space invader*, in the reverse-time setup) [156]. This exception arises due to the fact, that uniqueness of the solution can be established only *locally* in some particular cases. Another example is a *Norton's dome* [144], where a particle rests on the apex of a half-sphere, demonstrating the breakdown of solution's uniqueness when the force acting on the particle fails to satisfy Lipschitz condition. (For more detailed discussion on such exceptions see [58].) Again, these exceptions pertain to classical mechanics understood formally rather than to behaviour of realistic systems that require completion either by special relativity (particle's speed is limited by the speed of light) or quantum mechanics (initial conditions must account for quantum-mechanical degress of freedom). It is worth noting that the evolution described by Schrödinger equation (1.3) tends to be inherently *more deterministic* than that described by Newton's equations (for quantum analogues of above pathological examples see [57].)

Hence, we can clearly delineate between "epistemic" randomness, stemming from our incomplete knowledge, and "ontic" randomness, that refers to inherent property of the system [25]. In classical systems, the apparent randomness largely arises due to "practical" constraints and precise predictions are attainable through deterministic laws. To illustrate, consider a controlled coin-toss experiment where the outcomes can be effectively managed [168]. In other words, randomness in classical systems is intricately complementary to computational resources employed, hence can be regarded as relative. Consequently, it is the measurement procedure that distinguishes quantum mechanics from the above. As discussed in Chapter 2, while non-interacting quantum systems evolve deterministically according to Schrödinger's equations, the situation undergoes a change upon measurement, with outcomes following Born's rule (2.1)) and an irreducible event's probability is expected in general. We emphasize again that according to results such as Bell's inequalities, Kochen–Specker and other no-go theorems, these probabilities cannot be interpreted as (reduced to) incomplete, uncertain or biased knowledge, that otherwise could be overcome (cf. Chapter 2). These observations lead us to conclude that quantum-mechanical systems are natural candidates to embody intrinsic randomness in the real world.

To show that the concept of randomness is beyond mere academic interest, it is crucial to explain the significance of randomness in practical, real-world applications. Indeed, random phenomena are often perceived as undesirable attributes: noise can disrupt experiments, uncertainties necessitate the use of error analysis etc. From the technological standpoint, the general objective is to minimize random behaviour, maximizing control over experimental conditions. On the other hand, it is widely recognized that modern information technology, including domains such as cybersecurity, heavily relies on random number generation, where an inadequate "quality" of randomness may cause a lot of damage to both individuals and institutions (see [130]). To appreciate the advantages offered by randomness generated through quantum mechanics, it is beneficial to revisit general operational principles of contemporary random number generators (RNGs). Then it will be easy to distinguish them from the purportedly secure random number generation capabilities of quantum mechanics.

$$f(r),\ f(f(r)),\ f(f(f(r))) \text{ etc.}$$

In principle, the set S is finite due to memory limitations, what makes f a finite-state machine. The function f should possess sufficient non-linearity while still allowing for straightforward computation. As f is finite-state, its period (i.e. the minimum number of steps required to cycle back to a previous value) has to be finite; for instance, the most commonly used PRNG today, known as Mersenne's Twister, has a period of $2^{19937} - 1$, what virtually prevents f to repeat at all in a foreseeable future [132].

By this definition, it becomes evident that PRNGs are entirely deterministic, i.e. given a particular seed, the generator's output will consistently remain the same. Clearly, such a software-based procedures introduce a potential vulnerability, as one could attempt to deduce the generating function from a sequence of numbers. Therefore, the true challenge lies in devising a function that is virtually impossible to predict based on segments of its values. Unfortunately, the history is full of cyberattacks exploiting vulnerabilities in PRNGs [98, 121]; for an in-depth survey cf. [130]. This everlasting race could perhaps be commented by a quote by von Neumann: "(...) anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin" [143]. Nevertheless, as more and more powerful cryptographic tools are developed, we currently seem to be relatively safe with the prevalence of PRNGs. Still, this may evolve in future, as the field of quantum computing revolution becomes more mature and will start to challenge classical encryption algorithms.

Remark 33. Note that the well-known potential for circumventing "randomness" by using a more powerful machine is yet another illustration of "relative randomness". In the same spirit, the issue of epistemic randomness in classical physics was raised, where this relativity was strictly connected to purely practical limitations (see Remark 31). As we will see, this relativity phenomenon also manifests itself in the formal quantum-mechanical randomness, although overcoming it appears to be prohibited by the very structure of quantum mechanics.

So far, we have employed the notion of randomness in relatively loose and informal manner, with particular emphasis on number generation. However, we will see that as one tries to pin it down more precisely, the more elusive it appears. Let us discuss various approaches to define randomness in a mathematically precise manner in the next section.

6.2 What is randomness, precisely?

Let us temporarily set aside sequence generation processes and direct our attention to the outcomes they produce, namely numeric sequences (for simplicity, w.l.o.g. we will refer mainly to binary strings). Now, suppose someone hands us a (finite or infinite) sequence of numbers and asks a question: "Is this sequence random?". Naturally, we have certain intuitions here, and confronted with two sequences like

$$11001001000011111101101010100\ldots \quad \text{and} \quad 0000111100001111000011110000011\ldots$$

we would easily identify the second sequence as a regular, non-random one, while the first one would seem to devoid of any structure whatsoever. In fact, the seemingly random digits on the left are just initial digits of binary expansion of π and these are perfectly deterministic, with a closed-form formula for calculating successive members. This elementary observation challenges our common-sense perception of randomness, and calls for more sophisticated methods. Unfortunately, basic probabilistic intuition will not help us in this context: given any sequence $\sigma \in 2^{<\omega}$ of length $|\sigma|$, the probability of σ resulting from a perfect coin-toss follows the uniform probability distribution given by $P(\sigma) = 2^{-|\sigma|}$. This situation persists even when dealing with the space 2^{ω}, as now all sequences have a probability 0 under the uniform distribution. Consequently, although it appears that non-random, regular outcomes should show up less frequently, it cannot be supported by individual probabilities alone.

These simple observations reveal two crucial issues: firstly, one needs to carefully examine the "hallmarks" of randomness, allowing us to distinguish sequences that are "sufficiently random" from those that are not. Secondly, given that heuristically all sequences seem equally likely, it might be a good idea to certify the randomness through some underlying process. In other words, a sequence would be claimed random if it was generated exclusively by process, whose randomness is guaranteed by fundamental laws. Quoting von Neumann again: "There is no such thing as a random number — there are only methods to produce random numbers." [143]. Similarly, A. Khrennikov proposed to regard randomness as an entirely physical process, which cannot be defined in a mathematically consistent way [101]. In the following we elaborate on the problems making randomness so hard to define rigorously.

Recall that the probabilistic nature of quantum phenomena stems from both experimental evidence (consider an electron with spin up along the x-axis and then measuring its spin along the y-axis) and the underlying mathematical structure (cf. Born's rule in Chapter 2). In particular, the impossibility of the existence of local hidden variables, as discussed previously, agrees with the violation of the principle of sufficient cause [102] (namely, it might be impossible to point out the reason for a particle to have e.g. a definite spin in either direction). Moreover, we discussed how the results of measurement fall withing Born's rule, giving therefore at least a probabilistic flavour to quantum effects. However, the path that connects these probabilistic effects with a rigorously defined notion of random sequences is not as straightforward as one would initially assume. In fact, there is no single, universally accepted definition of randomness, what we briefly describe now.

Throughout history, three primary approaches have emerged for defining randomness. Namely, a random sequence σ should possess the following attributes: [118][101]

1. be incompressible (or sufficiently complex), implying that it should be impossible to provide a recipe for generating σ that is shorter than σ itself. This is mostly related to the work of Kolmogorov and his followers [123];

2. be unpredictable, meaning that there should not exist a martingale that succeeds on σ;

3. omit all specific characterizations or, put differently, devoid of patterns; here we refer mainly to works by Martin-Löf and others [55].

Remark 34. It is worth noting that the characteristic of random sequences that appears most crucial for practical applications is their unpredictability; the efforts of designing statistical tests and studying the complexity are made mainly due to their usefulness in predicting actual digits. Cryptography research on randomness emphasizes this even further, as nearly all algorithms aim to prevent the prediction of consecutive digits.

Before we will go into details of randomness pictures, let us review some fundamental facts regarding binary sequences, which will ultimately be interpreted as the outcomes of experiments in quantum mechanics. First, recall that given a topological space X, we define the Borel algebra $\mathrm{Bor}(X)$ to be the smallest σ-algebra that contains all opens from X (here, a σ-algebra of subsets of X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $X \in \mathcal{F}$, if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$ and for every countable collection $\{A_n\} \subseteq \mathcal{F}$ we have $\bigcup A_n \in \mathcal{F}$). Further we define a probability measure by $\mu : \mathrm{Bor}(X) \to [0,1] \subseteq \mathbb{R}$ by $\mu(X) = 1$, $\mu(\varnothing) = 0$ and $\mu\left(\bigcup A_n\right) = \sum \mu(A_n)$ whenever the elements of $\{A_n\}$ are pairwise disjoint. We call a subset $A \subseteq X$ to be measurable if there exists $B \in \mathrm{Bor}(X)$ such that the symmetric difference $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is null, i.e. $\mu(A \triangle B) = 0$.

The simplest $\{0,1\}$-sequences are the elements of the so-called Cantor space $2^\omega = \{\sigma : \omega \to \mathbf{2}\}$, and 2^ω is equipped with topology defined by opens $[\tau] = \{\sigma \in 2^\omega : \tau < \sigma\}$, where $\tau \in 2^{<\omega}$ is a finite $\{0,1\}$-sequence. Here, $\tau < \sigma$ means that τ is a prefix of σ (equivalently σ extends τ). This makes 2^ω a Polish space with a product measure μ defined by

$$\mu([\sigma]) = 2^{-|\sigma|} \tag{6.1}$$

(recall that a topological space is called Polish if it is homeomorphic to a complete metric space with no isolated points). The expression (6.1) provides yet another reason why probability alone cannot be of much use in determining randomness, as one observes that all the opens with the same prefix length possess identical probabilities, regardless of how much the prefixes differ in terms of apparent randomness. Note that one may consider random sequences in terms of various structures: $\mathbb{R}$, $\omega^\omega = \{\sigma : \omega \to \omega\}$ and 2^ω just introduced. Indeed, these spaces not only have the same cardinality, but also share similarities in terms of their Borel structure. We call $f : X \to Y$ a Borel isomorphism if for every $B \in \mathrm{Bor}(Y)$ it holds $f^{-1}(B) \in \mathrm{Bor}(X)$. Then we have the following

Lemma 13. Let $f : X \to Y$ be a Borel isomorphism. Then $A \subseteq X$ is Lebesgue measurable (of first category, with Baire property, measure zero) iff $f(A) \subseteq Y$ is Lebesgue measurable (of first category, with Baire property, measure zero).

Furthermore

Theorem 27. The sets $2^\omega, \omega^\omega$ and $\mathbb{R}$ can be made into Borel isomorphic Polish spaces, that are homeomorphic up to countable sets.

Let us go back to three approaches to randomness, in which we follow mainly [118, 93] (for an in-depth discussion see [54, 55]). To explore the concept of *incompressibility* of a binary string $\sigma \in 2^{<\omega}$, we introduce the notion of (Kolmogorov) complexity of σ. First, it is essential to have a basic understanding of Turing machines: these can be thought of as mathematical models for executing computation according to a program. More precisely, a Turing machine T takes a binary string σ as input and either halts, producing an output $T(\sigma)$, or it runs indefinitely. A Turing machine U is called *universal* if for any Turing machine T there exists $\tau \in 2^\omega$ such that for any $\sigma \in 2^\omega$ it holds

$$T(\sigma) = U(\tau\sigma),$$

i.e. for each T, a machine U always finds a string τ that computes all inputs of T through U. Further, recall that the set A of finite binary strings is called *prefix-free*, if for any $\sigma, \tau \in A$, neither σ is a prefix of τ nor τ is a prefix of σ. Then, a prefix-free Turing machine is a Turing machine with a prefix-free domain. In order to treat also infinite sequences $\sigma \in 2^\omega$ in general, one defines Kolmogorov prefix-free complexity $C(\sigma)$ for $\sigma \in 2^{<\omega}$ by

$$C(\sigma) = \min\{|\tau| \,:\, U(\tau) = \sigma,\, U \text{ — universal and prefix-free}\}$$

i.e. it is the length of the shortest program run on a fixed, prefix-free universal Turing machine U, that computes σ. We say $\sigma \in 2^\omega$ is c-compressible if $C(\sigma) < |\sigma| - c$, and call σ *Kolmogorov random* if it is not c-compressible for any $c \in \mathbb{N}$ (i.e. one has $C(\sigma) \geq |\sigma|$). Finally, a sequence $\sigma \in 2^\omega$ is called *Kolmogorov–Chaitin random* if

$$\lim_{N\to\infty} \frac{C(\sigma|_N)}{N} = 1,$$

i.e. a Kolmogorov prefix-free complexity $C(\sigma|_N)$ of initial N digits of σ grows as fast as N when $N \to \infty$.

Next, we discuss the *unpredictability*. Informally, we may describe a binary string (sequence) σ as unpredictable if we cannot predict any digit of σ given the other digits. In other words, if one considers a digit prediction in terms of betting, a string σ would be called unpredictable if there were no successful betting strategy on σ. This can be made rigorous with the help of a martingale, i.e. a function $d : 2^\omega \to [0, \infty)$ such that for every $\sigma \in 2^{<\omega}$ the so-called averaging condition holds:

$$d(\sigma) = \frac{1}{2}\left(d(\sigma 0) + d(\sigma 1)\right),$$

where $\sigma 0, \sigma 1$ denotes the concatenation of σ with 0 and 1, respectively. Such a function represents a betting strategy in that $d(\sigma|_N)$ is a payoff after N bets according to strategy d.

We say that a martingale d succeeds on a set $A \subseteq 2^\omega$ if

$$\limsup_{N\to\infty} d(\sigma|_N) = \infty \text{ for all } \sigma \in A. \tag{6.2}$$

Clearly, if (6.2) holds for a singleton $\{\sigma\}$, it implies that one can successfully predict digits of σ using a strategy d, hence we naturally arrive at the following notion of randomness: it should be impossible to find a succeeding martingale against a truly random (unpredictable) sequence. Unfortunately, it was Ville who proved

Theorem 28. For any $A \subseteq 2^\omega$, there exists a martingale that succeeds on A iff $\mu^\infty(A) = 0$.

Particularly, above result excluded *all* singletons from being considered random, rendering the condition too restrictive. To address this, one possible approach is to restrict considerations to computably enumerable martingales [158] (recall that a subset $A \subseteq 2^{<\omega}$ is computably enumerable if it is a domain of some partial computable function, i.e. a function $f : 2^{<\omega} \to 2^{<\omega}$ for which a Turing machine T exists such that for every $\sigma \in \mathrm{dom}(f)$ we have $T(\sigma) = f(\sigma)$ and $T(\sigma)$ halts). By Theorem 29 we will observe that this is an appropriate way of defining randomness in terms of unpredictability.

Finally, we discuss the notion of *patternlessness*. When we encounter a sequence σ that is non-random, we suspect such σ is atypical in some way. In other words, there is usually a specific property σ is characterized by (think of $\sigma \in 2^\omega$ such that $\sigma(2^n) = 0$ for $n \in \mathbb{N}$). On the contrary, a random sequence σ should be as typical as it is possible, meaning there should be no distinguishing "property" that singles out σ. In the present context, typicality and randomness are to be considered measure-theoretically. A prime example of such a property is provided by the outcome of a fair coin flip. It can be shown that the expression

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sigma_i = \frac{1}{2} \tag{6.3}$$

is typical in 2^ω, i.e. for $A \subseteq 2^\omega$, that consists of all sequences satisfying (6.3), it holds $\mu^\infty(A) = 1$. Thus almost all coin flips turn out to be "fair" (random), and the non-random ones belong to the set $2^\omega \setminus A$ which represents a specific property, since $\mu(2^\omega \setminus A) = 0$. It seems natural, then, to extend this idea and define a sequence to be random if it does not belong to any measure-zero set T, referred to as a "test" here. Thus, random sequences should omit all such tests. However, this approach leads to a serious issue, as it implies no sequence σ can deemed be random at all. This is because for every σ we have $\sigma \in \{\sigma\}$ and $\mu^\infty(\sigma) = 0$ evidently. We can see it resembles the case of martingales (see Theorem 28) and requires a similar resolution, but we need to introduce a few definitions first.

Thus, a set $A \subseteq 2^\omega$ is called *Martin-Löf null* if there exists a uniformly computably enumerable sequence (recall that a collection of sets $U_0, U_1, \ldots$ is uniformly computably enumerable if $\forall n \ (U_n = \mathrm{dom}(f_n))$ for a uniformly partial computable collection $\{f_n\}$, i.e. for a partial computable function f such that $f(n,x) = f_n(x)$ for all $x \in 2^\omega, n \in \mathbb{N}$) $\{U_n\}$ of computably enumerable subsets $U_n \subseteq 2^\omega$ such that $\mu(U_n) \leq 2^{-n}$ and $A \subseteq \bigcap_n U_n$. Again, the set $T = \bigcap_n U_n$ is called a Martin-Löf test and we say $\sigma \in 2^\omega$ is *Martin-Löf random* if $\sigma \notin T$ for any Martin-Löf test T. Thus the collection of measure-zero tests is seriously restricted to countably-many collection of Martin-Löf tests. Moreover it holds that there exists a universal test U such that $\sigma \notin U$ iff σ is Martin-Löf random.

Having established the three fundamental types of randomness that can be rigorously defined, a powerful result demonstrates that they are actually equivalent: [118, 55]

Theorem 29. A sequence σ is Kolmogorov–Chaitin random iff it is Martin–Löf random iff it is Schnorr random.

Since 1-randomness remains "invariant" when transitioning through contexts described above, one should ask whether it is connected to the randomness of quantum mechanics in any way. We close this section with a reference to the main result of the work [118]. Observe that the generic way of generating binary sequence in quantum mechanics is to measure an observable represented by a projection P multiple times. Then, the Born measure μ_P on $\sigma_P = \{0,1\} = 2$ (see Chapter 2) can be extended to 2^ω according to the following general procedure. Let (X, μ) be a measure space and consider the cylindrical σ-algebra $\mathcal{S} \subseteq \mathcal{P}(X^\omega)$, generated by the sets of the form

$$C = \prod_{n=0}^{N} C_n \times \prod_{k=N+1}^{\infty} X \tag{6.4}$$

where $C_n \in \mathrm{Bor}(X)$, $n = 1, 2, \ldots, N$. The probability measure μ^∞ on sets of the form (6.4) is defined by

$$\mu^\infty(C) = \prod_{n=0}^{N} \mu(C_n),$$

and this measure naturally leads a unique extension μ^∞ on X^ω of the probability measure μ on X. Returning to quantum mechanics, consider $\frac{1}{2}$-spin as a demonstrative quantum system (cf. Example 1). First, we prepare the system S in a certain state; w.l.o.g. let S be in a state $\psi = |+\rangle$ i.e. a spin-up along x-axis. It can be shown that a subsequent measurement of the spin along z-axis will give equal, $\frac{1}{2}$-probabilities of obtaining either spin up or down along z-axis, what stems directly from (2.5). The extension of the Born measure μ_P on $\{0,1\}$ to μ_P^∞ on 2^ω is defined accordingly, and the fair coin-flip Bernoulli process is recovered by the following [118]

Theorem 30. The following procedures for repeated identical independent measurements are equivalent (as they give the same possible outcome sequences with the same probabilities):

1. quantum mechanics is applied to the whole run, described as a single quantum-mechanical experiment with a single classically recorded outcome sequence (note that applying quantum mechanics to the whole run (at once) means employing a measurement of a family of commuting operators $\underline{a} = (a, \ldots)$ defined on an infinite tensor product $\mathcal{H}^{\otimes\infty}$; the above equivalence stems from the identity between measures
$$\mu_{\underline{a}} = \mu_a^\infty,$$
2. quantum mechanics is applied to single experiments (with classically recorded outcomes), upon which classical probability theory takes over to combine these.

Hence, either of the aforementioned methods ensures that the Born probability μ_a for a single outcome extends to a Bernoulli process characterized by the probability μ_a^∞ on the infinite sequence of experiments. Note that 2^ω represents the infinite sequences of repeated measurements of an observable P. Here, the measurement repetition is understood once again as a triple (t, a, b) with a time step $t : \omega \to \mathbb{R}$, state preparation $a \in \mathbf{S}$ (corresponding to state-density operator) and $b \in \mathbf{Q}$ (corresponding to question-projection) (see Chapter 2). Thus we have

Corollary 9. The Born measure μ_P determines the infinite-product measure μ^∞ on the Cantor space 2^ω of measurement outcomes in quantum mechanics.

Crucially, one shows the following [118]

Theorem 31. Almost every binary sequence $\sigma \in 2^\omega$ is 1-random with respect to μ^∞.

In other words, we have

$$\mu^\infty(\{\sigma \in 2^\omega : \sigma \text{ is 1-random}\} = 1. \tag{6.5}$$

Equivalently, the set of non-1-random sequences has null measure (has probability 0), or there is a zero probability of obtaining a sequence of outcomes that is not 1-random.

Remark 35. As emphasized in [118], it is important to distinguish here the probability of an outcome and of a property. In particular, for every outcome $\sigma \in 2^\omega$ we have $\mu^\infty(\sigma) = 0$, irrespective of whether σ is random or not (as we discussed at beginning of this chapter). However, (6.5) still remains valid for a property of 1-randomness.

6.3 Another level of randomness

Here we advocate for the concept that quantum mechanics may demand yet another, distinct approach to randomness, where a random sequence should be characterized by its *genericity*. This arises due to the nontrivial forcing extensions, inherent in the mathematical structure of quantum mechanics, described extensively in Chapter 2. In particular, recall that in [23] it was demonstrated that there exists a countable, transitive ZFC model M_0 capable of expressing the formalism of quantum mechanics, although the strong randomness of measurement outcomes rules out M_0 (as well as any forcing extension $M_0[G]$ of M_0) from being a *complete* description of quantum mechanics. (Note that Benioff's definition of randomness is tied with definability, which mirrors Gödel's constructible universe L_α.) In the subsequent discussion, we extend this approach and assert that any random sequence $\sigma \in 2^\omega$ of experiment outcomes should omit Borel-measure zero sets "coded" in the model M in which quantum mechanics is formalized. Consequently, these sequences live rather in a generic extension $M[\sigma]$, and it becomes apparent that there is a natural distinction between finite- and infinite-dimensional cases. The former extends Martin–Löf 1-randomness to the ω-randomness and the latter corresponds to the Solovay generic randomness. Moreover, we will argue that Boolean valuation of truth, inevitable along any forcing extension, suggests a modification of a measurement formulation due to von Neumann, see (2.3)-(2.4).

6.3.1 Finite-dimensional case

Recall that 1-randomness of quantum mechanics is a consequence of the way Born measure acts on the Cantor space 2^ω [116]. Based on this result, we will show that the formalism of quantum mechanics is actually not limited to 1-randomness. In order to extend it to n-randomness and further to ω-randomness, we need a couple of definitions. Recall that a sequence $\sigma \in 2^\omega$ is Martin–Löf null if σ passes all Martin–Löf tests and in general, σ is called n-random σ passes all ML_n tests (moreover, we say σ is *arithmetically random* if it is n-random for all $n \in \mathbb{N}$). To fix the notation we propose the following

Definition 4. Let T be a theory that assigns probabilities to outcomes of experiments via some probability measure μ. We say T is *n-random* for $n \in \mathbb{N}_+$, if this probability measure extends uniquely onto a probability measure on the set of infinite sequences of outcomes and additionally, almost all these sequences are n-random. If above is true for all $n \in \mathbb{N}$, we call T ω- (*arithmetically*) random.

Example 10. By Corollary 9 and Theorem 31, quantum mechanics is a 1-random theory, since the Born measure extends uniquely to 2^ω and almost all infinite sequences of outcomes in 2^ω are 1-random.

To show the n-randomness of QM, we first prove the following

Lemma 14. For every $n \geq 1$ it holds

$$\mu^{\infty}(\{\sigma \in 2^{\omega} : \sigma \text{ is } n\text{-random}\} = 1.$$

Proof. It is an immediate consequence of the Definition 4 of n-random sequences, namely that they omit all measure-zero sets that belong to the n-th arithmetic class and equivalently, every n-random sequence belongs to a full-measure subset of 2^{ω}. □

Theorem 32. QM is n-random for every $n \geq 1$, .

Proof. Let μ^{∞} be a unique extension of the Born measure, whose existence is guaranteed by Corollary 9. Based on Lemma 14, the set of all non-n-random sequences of quantum mechanical outcomes has null measure. □

By Definition 4 we conclude

Corollary 10. QM is ω-random.

Finally, we make a connection between randomness defined through arithmetic sets and Solovay genericity. First we define a sequence $\sigma \in 2^{\omega}$ to be *weakly n-random* if σ is an element of every n-arithmetic set of measure 1. Interestingly, this notion of randomness of $\sigma \in 2^{\omega}$ falls right in between n- and $(n-1)$-randomness as the following implications hold [95]

$$\sigma \text{ is } n\text{-random} \implies \sigma \text{ is weakly } n\text{-random} \implies \sigma \text{ is } (n-1)\text{-random} \tag{6.6}$$

Additionally we have

Theorem 33. A set is Solovay n-generic iff it is weakly n-random.

Observe that since QM is n-random for any $n \in \mathbb{N}_+$, by (6.6) it is also weakly n-random for any $n \in \mathbb{N}_+$. Defining ω-genericity in analogy to ω-randomness, by Theorem 33 we obtain

Corollary 11. QM is Solovay ω-generic.

In order to proceed with infinite-dimensional case, we make an important remark about Solovay n-genericity [95]. So far we have met a characterization of n-randomness as a property of omitting n-arithmetic sets of null measure. In fact, this can be shown to be equivalent to omitting null sets definable in the step L_{ω} of building a Gödel's constructible universe L (see Appendix B). Let ϕ be a sentence in the language of Peano arithmetic and let $\sigma \in 2^{<\omega}$; we say σ *forces* ϕ (written $\sigma \Vdash \phi$, as usual) if $A \models \phi$ for every $A \in \{\tau \in 2^{\omega} : \sigma < \tau\}$. Then for general $A \in 2^{\omega}$ we have $A \Vdash \phi$ if for some $\sigma \subset A$ it holds $\sigma \Vdash \phi$. Finally [95]

Proposition 3. *A is Solovay n-generic iff for each σ^0_n or Π^0_n sentence ϕ we have $A \Vdash \phi$ iff $A \models \phi$.*

Thus, one refers to the above as a *miniaturization* (effectivization) of set-theoretical Solovay forcing.

6.3.2 Infinite-dimensional case

Once we conclude that above description of randomness relies upon hereditarily finite sets of L_ω, we have good reasons to anticipate an enhancement of the approach in the case of infinite-dimensional Hilbert space $\mathcal{H}$. Firstly, L_ω is a ZF model without the axiom of infinity, what suggests heading towards full-fledged, countable transitive ZFC models (if not a universe V, eventually). Secondly, Boolean subalgebra $B \subseteq \mathcal{L}(\mathcal{H})$ can now be atomless, what is impossible in finite dimensions. This fact suggests a set-theoretical non-trivial forcing comes into play (see Lemma 21 for a fundamental importance of atomicity of B). Thirdly, this picture emphasizes the role of homomorphisms $h : B \to 2$ as a two-valued, classical reduction of a quantum-mechanical description. Suppose that such a reduction h is M-completely additive, i.e. for any $A \subseteq B$ such that $\sup A$ exists in B it holds

$$h(\sup A) = \sup\{h(a) : a \in A\} \quad (\text{in } M).$$

Then we have the following [166]

Lemma 15. A Boolean homomorphism $h : B \to 2$ is M completely additive iff $U = h^{-1}(1)$ is a generic ultrafilter over M in B.

But due to Lemma 21 we already know that a generic ultrafilter of a Boolean algebra is absent in M if and only if B is atomless in M. Therefore we conclude

Corollary 12. If $M = L_\alpha$ and B — atomless, then $h : B \to 2$ and $U = h^{-1}(1)$ are not in L_α. Moreover $U \in L_\alpha[U]$ is the random Solovay extension of L_α.

Observe that the formalism of quantum theory (i.e. an orthomodular lattice of projections together with the family of its maximal Boolean subalgebras) dictates that the classical reduction to two-valued logic has to go via random forcing extensions, namely choosing an observable a (or a family of compatible observables $\{a_i\}_{i\in I}$) to be measured, one effectively selects a (non-unique in general, see Example 8) maximal Boolean subalgebra B containing a (or $\{a_i\}_{i\in I}$). This choice gives rise to a specific Boolean-valued model L_α^B and generally to the family $\{L_\alpha^B\}_B$ indexed by $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$. The choice of an L_α-completely additive homomorphism $h : B \to 2$ then boils down to the choice of the ultrafilter U such that L_α^B/U is a genuine two-valued ZFC model (a random forcing extension $L_\alpha[U]$).

Remark 36. Up to this point, we have implicitly assumed that our discussion takes place within a countable transitive ZFC model (specifically L_α) and its extensions. For a complete picture we need to comment on the premises that quantum mechanics could be formulated in the universe of sets V (see Appendix B) instead. Suppose that quantum mechanics is Solovay generic random and it is formulated in V. In particular, there exists a random $\sigma \in 2^\omega$ that omits all null sets in V. Therefore one needs to consider an extension of V that contains σ. As V contains already all sets, the possible way out is to build a Boolean-valued model V^B inside L_α and find an ultrafilter U in B such that $V^B/U = V[\sigma]$. This reduces in fact to working with a countable transitive model again, what indicates one should consider the latter instead of V to formulate QM.

Recall that, according to Benioff [24], given countable transitive ZFC model M there is no Cohen extension $M[\sigma_C]$ that contains a random σ_C. We reformulate this fact in the case of Solovay generic randomness. Let $R(M), C(M)$ be the sets of random and Cohen reals, generic over M, respectively. Let $\mathcal{N}$, $\mathcal{M}$ be ideals of null and meagre Borel sets. Then, $\mathcal{N} \cap M$ and $\mathcal{M} \cap M$ represent null and meagre Borel sets coded in M, respectively. Likewise, $2^\omega \cap M$ consists of all real numbers in M; replacing M for any extension defines real numbers in that extensions. To obtain the result, first we state [114]

Lemma 16. Let $r \in R(M)$, $c \in C(M)$. It is provable in $M[r]$ that $2^\omega \cap M \in \mathcal{M}$ and $2^\omega \cap M \notin \mathcal{N}$. Also, it is provable in $M[c]$ that $2^\omega \cap M \in \mathcal{N}$ and $2^\omega \cap M \notin \mathcal{M}$.

Thus we obtain a variant of Benioff's result (see Chapter 2):

Corollary 13. For any Cohen extension $M[c]$ we have $R(M) = \varnothing$, hence no Cohen extension contains any random Solovay real.

Let us now discuss in more detail how random forcing affects the measurement procedure (for the background on measurement, see Chapter 2).

Remark 37. A toy model for representing logical independence measurement on appropriately prepared quantum state was proposed in [145]. In particular, it was noted that the axioms can be materialized as qubits' states; furthermore, if a measurement representing proposition gives a random outcome, then this proposition is logically independent of the axioms.

Above remarks suggest the following definition. We will call a quantum measurement of an observable a in a state ψ *generic* if $(\psi, \phi) \neq \pm\|\psi\| \cdot \|\phi\|$, where ϕ is a state ψ transformed by the measurement (in other words, the measurement is generic if states before and after measurement do not belong to the same ray in a Hilbert space).

Lemma 17. Any generic measurement of a quantum system in a state that belongs to infinite-dimensional Hilbert space H determines a pair (B, U), where $B \subseteq \mathcal{L}(\mathcal{H})$ is the measure algebra and $U \subset B$ is an ultrafilter.

Proof. Let a be any observable to be measured; since $\dim(\mathcal{H}) = \infty$, a is contained in a maximal, atomless Boolean algebra $B \subseteq \mathcal{L}(\mathcal{H})$. By Lemma 6, B is the measure algebra. Observe that any generic measurement, as defined above, provides a classical context that decides truth of all propositions within B via Theorem 18, given by the valuation $h : B \to \{0, 1\}$ that is completely additive. Therefore by Lemma 15 the set

$$h^{-1}(1) = U \subset B$$

determines a generic ultrafilter. □

Above result can now be relativised to any countable transitive ZFC model M via

Lemma 18. Let M be a countable transitive model of ZFC such that L_α is a submodel of M. Then any generic measurement of a quantum system in a state that belongs to infinite-dimensional Hilbert space H determines the pair (B, U) as in Lemma 17 (in M).

Theorem 34. Every generic measurement of a quantum system in a state that belongs to infinite-dimensional Hilbert space H gives rise to the classical realm by adding a random infinite sequence to the minimal countable transitive model of ZFC.

Proof. Once the measurement is done and the classical realm is attained, a specific valuation $h : B \to \{0, 1\}$ is defined, where B is a maximal Boolean context. Let ψ_{psc} be a pseudoclassical state (see Chapter 2), corresponding to a generic ultrafilter on the measure algebra B [171]. Therefore after the measurement, a valuation $h_{\psi_{psc}} : B \to \{0, 1\}$ given by an ultrafilter ψ_{psc} appears, such that

$$\psi_{psc} = h^{-1}_{\psi_{psc}}(1).$$

Now we relativise the above to a countable transitive ZFC model M. Then $\psi_{psc} \notin M$ and $\psi_{psc} \notin M[\psi_{psc}]$. Recall that every generic ultrafilter corresponds to a random real $r \in M[r] = M[\psi_{psc}]$ [91]. Such an r can be considered a random binary sequence with $M = L_\alpha$. □

6.3.3 Intermediate Boolean mixture of states

Recall that the orthodox formulation of measurement (2.3), (2.4) involves the interaction of a quantum system and a measuring device, which results in a mixed state. In the following, we will argue that this scheme requires an additional stage of a Boolean mixture of states by the following reasoning:

- Let M be a countable transitive ZFC model that formalizes quantum mechanics, therefore containing the complete measure algebra $B \subseteq \mathcal{L}(\mathcal{H})$.
- For quantum mechanics formulated in M, Solovay forcing leads to random reals $r \notin M$ where $M[r]$ is a Solovay forcing extension.
- An extension $M[r]$ can be obtained via Boolean-valued model M^B as

$$M[r] = M^B / U_r,$$

  where U_r is the ultrafilter corresponding to r.
- If $M = L_\alpha$, in order to attain random reals in 2-valued forcing extensions $\{L_\alpha : r \text{ — Solovay generic over } L_\alpha\}$, one has to go through the Boolean-valued model L_α^B.

The last observation above indicates that all *generic* measurements of a quantum system formalized in L_α factor through L_α^B to reach a 2-valued, classical stage. We propose a hypothesis that (2.3) should be replaced by a more general stage

$$\sum c_i |\phi_i\rangle |a_i\rangle \in L_\alpha^B \xrightarrow{\text{forcing}} L_\alpha \ni |\phi_n\rangle |a_n\rangle \tag{6.7}$$

Let us show

Lemma 19. Suppose quantum mechanics is formalized in L_α. Then the Boolean stage L_α^B in a generic measurement is equivalent to the existence of random forcing extensions.

Proof. It is easy to see that L_α^B determines all random extensions L_α simply by L_α^B / U_r where U_r is a generic ultrafilter corresponding to random r; conversely, every random forcing extension L_α comes from L_α^B via L_α^B / U_r. □

Since random forcing extensions correspond to generic measurements by Theorem 34, we obtain

Corollary 14. A generic measurement of a quantum system involves the intermediate Boolean stage described in (6.7).

Before we discuss experimental aspects of quantum randomness, let us briefly comment the relationship with hidden variables (see Chapter 2). Recall that hidden variable theories primarily introduce additional degrees of freedom to make a theory deterministic. Thus, going back to mathematical approaches to randomness, we identify that hidden variables align with predictability interpretation (see Section 6.2). Therefore one might expect that hidden variables and randomness are fundamentally opposed. As shown earlier, quantum theory formulated in L_α provides that a classical two-valued realm in the context of some maximal measure algebra $B \subseteq \mathcal{L}(\mathcal{H})$ is obtained via a completely additive homomorphism $h : B \to \{0, 1\}$ and random generic extensions dictated by $h^{-1}(1)$, which corresponds to the preimage of what is considered true in the extension. If a genuine hidden variable theory existed, it would have to admit generic extensions for *all* local contexts $B \subseteq \mathcal{L}(\mathcal{H})$. In other words, for a hidden variable theory to be valid, there should exist a countable transitive ZFC model containing this particular realization of hidden variables. This is not the case as shown by

Lemma 20. There is not any countable transitive ZFC model containing all random extensions of L_α.

Proof. Let M be a countable transitive ZFC model. In particular, M contains only countably many extensions of L_α. Since there is uncountably many extensions of L_α, there is also uncountably many of them outside M. □

Although the above is in agreement with the standard results such as Kochen–Specker theorem, it appears to be extremely excessive. In fact, due to Kochen–Specker theorem, it is sufficient to assume $\dim(\mathcal{H}) \geq 3$ and *at least* two incompatible contexts in order to prohibit global valuations. This raises questions about the interpretation of Lemma 6.3.3. Fortunately, it is possible to address finite number of contexts. Firstly, a weaker version of can be stated as follows: there exist two different random forcing extensions $M_1 = L_\alpha[U_1], M_2 = L_\alpha[U_2]$ such that there is no $M_3 = L_\alpha[U_3]$ with M_1, M_2 both being submodels of M_3. This restated version still rules out hidden variables, and it is expressed precisely by the following result [77].

Proposition 4. Let M be a countable transitive ZFC model and let $M_1 = M[U_1]$ be its random extension. Then there always exists $M_2 = M[U_2]$ such that there does not exist any random extension $M[U]$ with the same ordinals and containing M_1, M_2 as submodels.

Corollary 15. Since every random extension $L_\alpha[U]$ preserves ordinals, hence Proposition 4 applies here and it follows that it is sufficient to have at least two incompatible Boolean contexts to forbid hidden variables.

6.4 QRNGs and infinite-dimensional systems

We conclude this chapter with a discussion on the possibility of experimentally testing the randomness of quantum mechanics. We have previously argued that determining the randomness of a given binary sequence, generated by some (possibly hidden) algorithm, is not a feasible task, as it is generically an incomputable procedure. The example of π can be taken to the extreme by considering a random sequence restored from some hardware memory, necessarily turning the sequence into a predictable one [25]. This implies that common tests of randomness, like relative frequency or Borel normality, establish only a prerequisite for the sequence to be random. In other words, these tests reveal the degree to which a sequence can be considered random and identify sequences that are *likely* to be non-random. Moreover, according to Ramsey's theory, even an infinitely random sequence could theoretically exhibit patterns or correlations, potentially failing specific statistical tests in principle [42]. To address this point, it is reasonable to shift the focus towards the generation process. As a result, ensuring the randomness of a numeric sequences has led to an emphasis on physical processes that generate these and "certify" a certain degree of randomness. For instance, expriments involving photons passing through a polarization beam splitter have produced sequences of seemingly unpredictable, random outcomes [92]. Note that measuring the two-dimensional state (such as a state of a photon with two polarizations) can be easily represented by binary sequences. However, the only formal basis for such a sequence to claim its randomness is again Born's rule. To establish a more rigorous theoretical foundation, attention turns to principles that certify unpredictability more firmly. In particular, we focus here on two aspects: Kochen–Specker theorem and Bell inequalities (cf. Chapter 2). Recall that Kochen–Specker theorem asserts that, for $\dim(\mathcal{H}) > 2$, it is impossible to assign in advance the results of measurement in a noncontextual way, i.e. the one that is independent of measurement.

Consequently, a carefully designed setup of such a quantum system can theoretically provide a certified method for generating randomness. Due to dimensional constraints, the experimental realization discussed in [113] relied on a qutrit (spin-1) system, which is a three-dimensional analogof the more familiar spin-$\frac{1}{2}$ qubit. Importantly, a constructive version of the Kochen–Specker theorem has been demonstrated, implying that in order to find specific observables with indefinite values, one must identify one-dimensional projections for which the system is not in an eigenstate [1]. Recently, it has been rigorously demonstrated that, based on the above indefinitess criterion and the qutrit experimental setup, measurements yield indefinite results, when performed on the system in a certain, prepared eigenstate [3]. Moreover, these results can be shown to be unpredictable (i.e. there is no computable function returning those outcomes). Recall that Bell's inequalities impose constraints on correlations between measurement outcomes in the context of a local, hidden-variable theory. When a quantum system consists of two sufficiently separated subsystems, the violation of Bell's inequalities serves as another means of certifying randomness. In particular, it has been proposed that ions trapped in vacuum chambers that emit photons can give rise to such subsystems; entangling the photons in a beamsplitter results then in the entanglement of the ions, and the detection of photons in appropriate basis yields the measurement of two-state (qubit) ion systems [146]. The observed violation of Bell's inequalities precisely reflects the fact, that there is no deterministic process describing the series of measurements (in other words, the outcomes could not be predetermined in advance).

Let us now shift our focus to the so-called a posteriori tests, which represent a more conventional approach to randomness testing. Again, the first remark is that an algorithmic randomness of a sequence, as understood in terms of incompressibility, is not effectively decidable. One way out is to carefully design a test that explores an algorithmic randomness as much as possible, and then compare the results with those obtained from chosen PRNGs. In [2], the authors examined the qutrit measurement results described above, in comparison with five deterministic, pseudo-random sequences. The first test for Borel normality revealed a significant bias in the case of a quantum sequence; this, although expected, can be postprocessed in principle e.g. by the so-called von Neumann's trick [113]. Following this, four tests aimed at investigating algorithmic randomness (so-called Chaitin–Schwartz–Solovay–Strassen tests) were applied. While the initial hope was to provide the clear distinction in computability between truly random and pseudo-random sequences, the results were not conclusive: the tests either proved no significant difference, or, in case of differences, they were likely due to the mentioned earlier bias. Recently, slightly more optimistic, but similar results have been demonstrated [96]. This also suggests a potential room to investigate other tests for algorithmic randomness in the future.

To summarize, algorithmic randomness of QM is currently verified within two main steps: first, we certify the randomness on theoretical, formal grounds (as the Kochen–Specker theorem or Bell-type inequalities); second, we verify the sequence outcomes to pass through the tests all patternless sequences should pass. While the former does not raise serious doubts and gets different formulation under our work (namely the randomness is to be certified by passing through ZFC models, that occurs during the measurement process), the latter gives usually ambiguous results, that should be also addressed thoroughly in the future from the point of view of algorithmic randomness described in the present work.

In the end, let us comment on dimensional aspects of experimental randomness. Firstly, we distinguished between quantum systems represented by finite- and infinite-dimensional Hilbert spaces. Experimentally determining whether quantum randomness is enhanced and originates from an infinite-dimensional system can be challenging. However, there is an alternative approach. Certain quantum constructions, such as those related to canonical commutation relations (see comments to axioms in Chapter 2) require the Hilbert space to be infinite-dimensional in order to be satisfied. Therefore, if a measurement on a quantum system S, resulting in a sequence $\sigma \in 2^{\omega}$, decided whether $\mathcal{H}_S$ is finite- or infinite-dimensional, it would provide an information on the strength of randomness of σ. Currently, main focus of such *dimension witnesses* lies on designing experiments that quantify finite dimensions [40, 39]. However, to obtain a lower bound on the dimension of a Hilbert space describing measured physical systems, a particular variant of Clauser–Horne–Shimony–Holt (CHSH) game has been designed. It has been shown that for appropriate entanglement between subsystems, there exist correlations that cannot be realized by *any* finite-dimensional system [48] (the results hold even for finite number of questions and outcomes). Unfortunately, these results cannot decisively distinguish between finite- and infinite-dimensional systems, as the calculated correlations can be approximated by systems with sufficiently large dimensions to arbitrary precision. Put differently, given a correlation value and the precision of experiment, there will be always a specific finite-dimensional setting that would fit a theoretical value with some measurement error. Similarly, tests of dimensionality based on the experimental verification of the uncertainty principle for position and momentum face limitations. Namely, there exist finite-dimensional canonical commutation relations (CCR) for sufficiently large dimensions that cannot be distinguished from the exact CCR realized in the infinite-dimensional setup [162]. This puts into question the possibility of actually detecting the infinite number of dimensions of a Hilbert space (however, further work on random number generators based on operators of continuous spectrum may shed some light on that issue, cf. [129]). Applying several tests simultaneously would be another possibility, as finite-dimensional approximations to various effects may behave differently. Although we cannot provide here any ready-to-go solution, we suggest that quantum correlations between regions in spacetime could elude the finite-dimensional approximation, therefore gravitational effects might reveal some possibilities of realizing such tests.

Chapter 7

Discussion and outlook

We have explored various applications of topos theory in quantum physics. Majority of discussed topics originate from the lattice structure $\mathcal{L}(\mathcal{H})$ and the family $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$ of its Boolean subalgebras. We have shown that it is sufficiently rich and it generates naturally a family of sheaf topoi $V^B = \mathrm{Sh}(B)$, defined through Boolean-valued models, which serve as an intermediate step to forcing extensions. The elements of $\mathrm{BSub}(\mathcal{L}(\mathcal{H}))$, derived from the components — projections in $\mathcal{L}(\mathcal{H})$ — determine extensions V^B/U. Through these extensions it has been established next how to reinterpret the contributions of zero-point energy of quantum fields. In particular, the adopted model makes the contributions vanishing, what provides a starting point for addressing the cosmological constant problem. At the same time, every topos V^B carries the object of real numbers $\mathbb{R}_B$ on its own. In the case of $B \in \mathrm{BSub}(\mathcal{L}(\mathcal{H}))$, real numbers $\mathbb{R}_B$ are exactly self-adjoint operators contained in B, what gives an interesting interpretational aspect of the approach. Trying to model globally spacetime as a smooth manifold $\mathbb{R}^n$ built upon the patches parametrized by such reals, one concludes that such a spacetime has to be an exotic smooth $\mathbb{R}^4$. This is treated then as a guiding principle in searching for a derivation of a small, non-zero value of cosmological constant that appears as a topological invariant and it is indeed possible to calculate a value that agrees with current results of experiments. The parametrization of the manifold's atlas by Boolean subalgebras is then shown to possess a similar categorical structure — the colimit — inside appropriate categories. The discussion has been completed with a recent result showing that the local modification of a smooth manifold by a Basel topos sheds new light on distribution theory, in which we briefly refer to the work [107] that offers another perspective on generating exotic smoothness by interpreting some of transition maps inside the Basel topos. Finally, based on the structure of $\mathcal{L}(\mathcal{H})$ once again, it was shown how quantum-mechanical randomness is affected by employing Boolean topoi to describe infinite-dimensional quantum systems; while it was known that in the case of finite-dimensional systems, quantum mechanics enjoys the property of 1-randomness, we have proved that atomless subalgebras of $\mathcal{L}(\mathcal{H})$ lead to the stronger notion of randomness, described generically in the language of Boolean-valued models and forcing. In the following, we would like to indicate several points in the thesis that appear promising for future exploration.

1. Chapter 3: As we started the discussion with the remarks concerning the structure of $\mathcal{L}(\mathcal{H})$ and its Boolean subalgebras in particular, recall that we only touched upon the groups of automorphisms defined over several (sub-)structures of $\mathcal{L}(\mathcal{H})$. Based on the relation between the algebraic structure of $\mathcal{L}(\mathcal{H})$ and spacetime's smooth structure M, it is reasonable to ask how the homorphisms of Boolean subalgebras relate to diffeomorphisms of charts on M and, globally, what is a relation between automorphisms of $\mathcal{L}(\mathcal{H})$ and global diffeomorphisms of M (this extends also to transformations between sheaf toposes V^B corresponding to "local" frames of reference [50]). This way we arrive at interesting point for future research, possibly building upon the work [110, 103]. Namely, recall that every Casson handle can be embedded

in a linear one, further represented by a real number. Consequently, if formulated in a countable ZFC model M, every non-trivial forcing extension $M[G]$ adds reals to the model, and these reals represent again some Casson handles that are missing in M. This opens a door to explore diffeomorphism classes of exotic $\mathbb{R}^4$ through set-theoretic tools. For example, it is not known when forcing extensions $M[G_1]$, $M[G_2]$ lead to (non)diffeomorphic Casson handles given by the respective ultrafilters G_1, G_2. Accordingly, it would be also interesting to provide interpretation for Proposition 4, as for every model M we find forcing extensions, hence some internal Casson handles, that are not diffeomorphic in any further extension. Finally, this is related also to automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ (see Chapter 3), as one can lift the diffeomorphism between Casson handles (Cohen reals) to almost permutations of $\mathbb{N}$. These define in turn so-called trivial automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ and it has been conjectured that non-trivial automorphisms would correspond to exotic smooth $\mathbb{R}^4$ giving rise to hypothetical exotic smooth S^4 [109].

2. Chapter 4: The necessary condition of an atomless Boolean algebra of projections to generate nontrivial forcing extension has the weakness as shown in Example 8. Since atomicity of a block containing an operator a is not an invariant of a, one would need some effective solution providing conditions for the block's atomicity. Also, suppressing zero-point modes of quantum fields requires a thorough analysis, including extending the discussion to other forcing notions that could be applied there. The value of cosmological constant obtained via topological transitions needs to be analyzed with respect to its stability under boundary conditions such as the size of an initial sphere, choice of homology spheres etc. Finally, as described in [13], an embedding of small exotic $\mathbb{R}^4$ into standard one (instead of $K3\mathbb{C}\bar{P}^2$) causes topologically-induced cosmological constant to vanish. An interpretation of such an effect certainly needs to be considered in terms of the modification of "local" real line as described in the beginning of Chapter 4.

3. Chapter 5: As the categorical structures for Boolean subalgebras of $\mathcal{L}(\mathcal{H})$ and opens in smooth atlas were identified and the global objects were shown to agree, it is a natural next step to explore whether the structures are related in a more fundamental way. In particular, one should investigate this relation can be made functorial, also in the context of the remark to Chapter 3 above. Considering a topos-theoretical description of smoothness through the Basel topos $\mathcal{B}$, it would be interesting to examine what particular exotic smooth $\mathbb{R}^4$ are produced via a local modification of transition maps, what would possibly lead to the construction of smooth invariants. Note that, although it was deliberate here to avoid description through Casson handles, there seems to be also a deep connection between the local modification of the manifold with $\mathcal{B}$ and infinite constructions in 4-dimensional differential topology (see also the comment to Chapter 3 above). This relationship operates through smooth objects of natural and real numbers inside $\mathcal{B}$, which allow to pass from infinite to s-(smoothly-)finite notions such as infinite signed trees that appear in exotic constructions [11]. Additionally, Remark 29 only briefly mentioned potential applications of interpreting some divergent quantities that appear e.g. in general relativity or in quantum field theory in $\mathcal{B}$ (see also [30]). Still, such re-interpretation of renormalization is not intended to be viewed as some "miraculous" tool to make divergences vanish, as these usually indicate the break-down of an effective picture. Indeed, we conjecture that making such quantities s-finite in $\mathcal{B}$ will produce procedures equivalent to standard ones, such as attaching the counterterms to scattering amplitudes. Most importantly, as described in Chapter 5, such a local modification

does not come for free and the price to pay is the modification in the smooth structure. This offers a new, qualitative perspective on the way renormalization relates to the background spacetime. A more quantitative examination could potentially reveal a direct relationship between concrete examples of renormalized quantum field theories defined on $\mathbb{R}^4$ and particular (exotic) smooth structures on $\mathbb{R}^4$.

4. Chapter 6: Here, the experimental distinction of randomness in the case of finite- and infinite-dimensional quantum systems is still unclear. As we argue, it is a common theme in physics and constructions dubbed as "infinite", "infinitely small", "smooth" etc. are never easy to grasp experimentally, mainly due to omnipresent measurement precision constraints. It is also interesting to develop the alleged correspondence between forcing extensions and many-worlds interpretation of quantum mechanics. Finally, from the perspective of decoherence caused by gravity bath, it is worth examining what possible impact spacetime (gravity) may have on randomness.

Appendix A

Lattices and Boolean algebras

Definition 5. A partially ordered set (poset) is a pair $(P, \leq)$ (written simply P if it not leads to confusion) such that P is a non-empty set and $\leq$ is a binary relation that is reflexive, antisymmetric and transitive, i.e.

1. $\forall p \in P\,(p \leq p)$;
2. $\forall p, q \in P\,\big((p \leq q \text{ and } q \leq p) \implies p = q\big)$;
3. $\forall p, q, r \in P\,\big((p \leq q \text{ and } q \leq r) \implies p \leq r\big)$.

We write written $p < q$ for $p \leq g$ and $p \neq q$. Moreover, a partial order $(P, \leq)$ is called linearly (totally) ordered if all elements are comparable, i.e. for all p, q either $p \leq q$ or $q \leq p$. Finally, we call $(P, \leq)$ well-ordered if it is totally ordered and every subset of P has the least element.

Definition 6. A lattice is a poset $(L, \leq)$ such that for any $p, q \in L$ there exist the least upper bound (supremum or join) $p \vee q$ and the greatest lower bound (infimum or meet) $p \wedge q$ in L. A lattice L is called

- bounded, if there is a minimal (bottom) element $0 \in L$ such that $\forall p \in L\,(0 \leq p)$ and a maximal (top) element $1 \in L$ such that $\forall p \in L\,(p \leq 1)$,
- distributive, if $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ for any $p, q, r \in L$.
- complete, if every subset X of L has a supremum and an infimum,
- orthocomplemented (i.e. L is an ortholattice) if L is bounded and equipped with an operation $\perp : L \to L$ that satisfies $p^{\perp} \wedge p = 0$, $p^{\perp} \vee p = 1$ and if $p \leq q$ then $q^{\perp} \leq p^{\perp}$,
- orthomodular if $p \leq q \to p = q \wedge p \vee q^{\perp}$ for any $p, q \in L$.

Remark 38. Equationally, a bounded lattice L can be defined by

$$
\begin{array}{cc}
a \vee 0 = a & a \wedge 1 = a \\
a \vee a = a & a \wedge a = a \\
a \vee b = b \vee a & a \wedge b = b \wedge a \\
(a \vee b) \vee c = a \vee (b \vee c) & (a \wedge b) \wedge c = a \wedge (b \wedge c) \\
(a \vee b) \wedge b = b & (a \wedge b) \vee b = b
\end{array}
$$

for any $a, b, c \in L$.

Example 11. Let $\mathcal{L}(\mathcal{H})$ be a set of projections on a Hilbert space $\mathcal{H}$. Then $(\mathcal{L}(\mathcal{H}), \leq)$ is a poset with

$$p \leq q \iff \operatorname{ran}(p) \subseteq \operatorname{ran}(q) \iff pq = qp = p.$$

Clearly, this order is reflexive, antisymmetric and transitive. Moreover, $\mathcal{L}(\mathcal{H})$ is bounded with 1 and 0 defined as projections on whole $\mathcal{H}$ and an empty subspace, respectively, since for any $p \in \mathcal{L}(\mathcal{H})$ it holds that $\varnothing \subseteq \text{ran}(p) \subseteq \mathcal{H}$. Define supremum and infimum as follows:

$$\bigvee P_i = P_M \text{ where } M \text{ is the closed linear span of } \bigcup \text{ran}(P_i),$$
$$\bigwedge P_i = P_M \text{ where } M = \bigcap \text{ran}(P_i).$$

Hence also $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ for any $p, q, r \in \mathcal{L}(\mathcal{H})$. Note that one can not take an ordinary union of closed subspaces to be the join, because the result is not a closed subspace in general. If projections $p, q \in \mathcal{L}(\mathcal{H})$ commute, i.e. $pq = qp$, above formulae for join and meet boil down to

$$p \vee q = p + q - pq, \quad p \wedge q = pq.$$

Notice that $p^{\perp} := 1 - p$ defines an orthocomplementation in $\mathcal{L}(\mathcal{H})$, making it an ortholattice. However, it does not always hold that $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$, hence $\mathcal{L}(\mathcal{H})$ is not distributive in general. Moreover, $\mathcal{L}(\mathcal{H})$ is orthomodular what makes it a complete orthomodular ortholattice.

Remark 39. Every OML can be represented as the lattice of closed subspaces (equivalently, by the lattice of projections on these subspaces) of some Hilbert space [86].

Definition 7. Let L be a lattice. An element $p > 0$ is called an atom if

$$\forall q \in L \; \left(q \leq p \implies (q = 0 \vee q = p)\right).$$

Definition 8. Let L be a distributive, bounded lattice. A filter on L is a set $F \subset L$ that is closed both upwards and with respect to meet, i.e.

1. if $p \in F$ and $q \geq p$, then $q \in F$,
2. if $p, q \in F$, then $p \wedge q \in F$.

Moreover, a filter $F \subset L$ is called

- *prime* if $\forall p, q \in L \, (p \vee q \in F \implies (p \in F \vee q \in F))$
- an *ultrafilter* if $\forall p \in L \left(p \in F \vee \neg p \in F\right)$

Observe that for a filter F, it always holds that $1 \in F$ and $0 \notin F$ (otherwise F would not be a proper subset of L). Another simple consequence is the fact that $p \in F$ entails $\neg p \notin F$.

Example 12. Any non-zero element $p \in L$ generates a particular filter $\{p\}^+ = \{q \,|\, p \leq q\}$, called a principal filter. It is easy to see that $\{p\}^+$ is the smallest filter containing p.

An ultrafilter U is called principal if there exists an atom $a \in L$ such that

$$U = \{b \in L \,:\, a \leq b\}.$$

Ultrafilters that are not principal are also called free.

Remark 40. The notion dual to filter is that of an ideal (i.e. it is a subset closed both downwards and with respect to join), hence all facts concerning filters can be dualized to statements about ideals.

Definition 9. A Heyting algebra is a bounded lattice H such that for any $a, b \in H$, a set

$$\{c \mid c \in H \text{ and } a \wedge c \leq b\}$$

has a largest element, denoted by $a \implies b$; an operator, that assigns $a \implies b$ to elements $a, b \in H$, is called an implication.

Remark 41. A well-known equivalence relation can be defined as an operation $\iff$ that assigns to each pair $a, b \in H$ an element

$$a \iff b := (a \implies b) \wedge (b \implies a)$$

Remark 42. Again, one can characterize Heyting algebras equivalently by equations that $\implies$ operation should obey. Namely, each Heyting algebra H is a distributive, bounded lattice with a binary operation $\implies : H \times H \to H$ such that

$$\begin{aligned} a \implies a &= 1 \\ a \wedge (a \implies b) &= a \wedge b \\ b \wedge (a \implies b) &= b \\ a \implies (b \wedge c) &= (a \implies b) \wedge (a \implies c) \end{aligned}$$

for all $a, b, c \in H$.

Example 13. Let X be a topological space and let $\mathcal{O}(X)$ be a collection of open sets in X. Then $\mathcal{O}(X)$ is a Heyting algebra with join $A \vee B$, meet $A \wedge B$ and $A \implies B$ given by sum $A \cup B$, intersection $A \cap B$ and $\mathrm{Int}\left(A^C \cup B\right)$ operations, respectively, where $A^C = X \setminus A$ is the set-theoretical completion. Moreover, it holds that $\mathcal{O}(X)$ is infinitely distributive in the sense of

$$O \cap \left(\bigcup_{i \in I} O_i\right) = \bigcup_{i \in I} (O \cap O_i)$$

for any $O \in \mathcal{O}(X)$ and $\{O_i\}_{i \in I} \subseteq \mathcal{O}(X)$.

The above example is in fact exhaustive due to the following [20]

Theorem 35. Any Heting algebra is isomorphic to an algebra of opens.

Remark 43. In Heyting algebras, the notion of a complement may be weakened to the so-called pseudocomplement, i.e. an operation that sends each $a \in H$ to the element $a^* := a \implies 0$. Note that pseudocomplement becomes a complement when it obeys the law of excluded middle, i.e.

$$\forall a \in H \ \left((a^*)^* = a\right) \iff \forall a \in H \ (a \vee a^* = 1)\,. \tag{A.1}$$

The fact that Heyting algebras do not necessarily satisfy (A.1) is the reason these algebras formalize intuitionistic propositional logic (cf. Appendix B for the internal logic in topoi). Before one eventually narrows the discussion down to Boolean algebras and classical logic, it is important to bring yet another structure, that will serve as a logical underpinning for a generalization of topological space (see Appendix B).

Definition 10. A frame is a complete distributive lattice such that

$$x \wedge \bigvee_{\lambda} y_{\lambda} = \bigvee_{\lambda} x \wedge y_{\lambda},$$

where $\{y_\lambda\}$ is not necessarily a finite set.

One can show that frames are actually complete Heyting algebras.

Example 14. Important and somewhat opposite examples of frames come from a topological space X endowed with its topology $\mathcal{O}(X)$ and a family of regular open sets $\mathcal{O}_{reg}(X) = \{U \subseteq X : \neg\neg U = U\}$, where $\neg U = \mathrm{int}(X \setminus U)$.

Definition 11. A Boolean algebra is a Heyting algebra that satisfies one (hence both) of equalities (A.1). Equivalently, a Boolean algebra is an orthocomplemented, distributive lattice. An implication in any Boolean algebra can be defined by $a \implies b = \neg a \vee b$.

Example 15. Let X be any set. Then $\mathcal{P}(X)$ is a Boolean algebra with meet, join and complement defined by $\cap$, $\cup$, $(-)^C$, respectively.

We call B atomic if for every $q \in B$ there is an atom p such that $p < q$; we call B atomless if it does not contain any atoms, and the same applies to lattices in general.

Example 16.

1. All finite Boolean algebras are atomic.
2. $\mathcal{P}(X)$ is always atomic and the atoms are singletons $\{x\}, x \in X$.
3. $\mathcal{L}(\mathcal{H})$ is always atomic and the atoms are one-dimensional subspaces $\{c\psi \mid c \in \mathbb{C}\}, \psi \in \mathcal{H}$.
4. The set $\mathrm{Bor}(\mathbb{R})/\mathcal{N}$ of Borel subsets of $\mathbb{R}$ modulo Lebesgue measure null sets is an atomless Boolean algebra.
5. The set $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ of all subsets of $\mathbb{N}$ modulo an ideal of finite sets is an atomless Boolean algebra.

One of the most remarkable facts concerning Boolean algebras is the following analogue of Theorem 35 [20]

Theorem 36 (The Stone Representation Theorem)**.** Any Boolean algebra is isomorphic to a certain field of sets.

Recall that a field of sets is just a subalgebra of a powerset of certain set. More specifically, every Boolean algebra B is isomorphic to the algebra of closed-open subsets of its *Stone space* $S(B)$, where $S(B)$ is the collection of ultrafilters on B, and the topology of $S(B)$ is generated by the clopen preimages of 1 in the trivial Boolean algebra $\{0, 1\}$). Conversely, every Stone space X can be turned into Boolean algebra $\mathrm{cl}(X)$ by taking all the clopen sets of X.

Appendix B

Categories and models

B.1 Categories

We start with a very brief reminder on the basic category-theoretic notions; also, we provide some illustrations whenever we feel it helps in grasping the content. We refer the reader to [21] for more detailed discussion.

Definition 12. A category $\mathbf{C}$ consists of a collection of objects $\mathrm{Ob}(\mathbf{C})$ and a collection of arrows $\mathrm{Arr}(\mathbf{C})$ such that

- to every $f \in \mathrm{Arr}(\mathbf{C})$ there are domain $\mathrm{dom}(f)$ and codomain $\mathrm{cod}(f)$ objects assigned; we note that by $f : \mathrm{dom}(f) \to \mathrm{cod}(f)$
- for every $f : A \to B$ and $g : C \to C$ there exists $g \circ f : A \to C$
- for every object A there exists an identity arrow $1_A : A \to A$ such that

  $$f \circ 1_a = f, \quad 1_a \circ g = g$$

  for every $f : A \to B$ and $g : C \to A$
- for every triple $f : A \to B$, $g : B \to C$, $h : C \to D$ it holds

  $$(h \circ g) \circ f = h \circ (g \circ f).$$

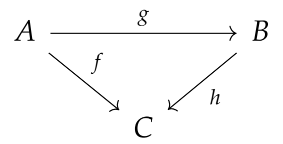
We denote $\mathrm{Hom}_{\mathbf{C}}(A, B) = \{f \in \mathrm{Arr}(\mathbf{C}) : \mathrm{dom}(f) = A, \mathrm{cod}(f) = B\}$. Given the category $\mathbf{C}$, the opposite category $\mathbf{C}^{\mathrm{op}}$ is a category that contains the same objects as $\mathbf{C}$, but all arrows of $\mathbf{C}$ have been reversed. Surprisingly, the above definition is general enough to include a lot of known mathematical structures, like the category of sets and functions **Set**, groups and homomorphisms **Grp**, topological spaces and homeomorphisms **Top** or smooth manifolds and smooth maps **Mfd**, frames and frame homomorphisms **Frm**, to name only a few. Moreover, one can make a category out of a single object; for example, a poset P may be viewed as a posetal category $\mathbf{P}$, where $\mathrm{Ob}(\mathbf{P}) = P$ and there is an arrow $p \to q$ (i.e. $\mathrm{Hom}_{\mathbf{P}}(p, q)$ is a one-element set) whenever $p \leq q$.

Definition 13. An arrow $f : A \to B$ is called

- an epimorphism (epic) if it is right-cancellable
- a monomorphism (monic) if it is left-cancellable
- an isomorphism (iso) if there is an inverse arrow in the category

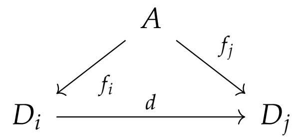
Again, illustrative instances of above properties come with **Set** with surjections as epics, injections as monics and bijections as isomorphisms.

Definition 14. A diagram **D** is a set of objects and arrows between these objects, e.g.



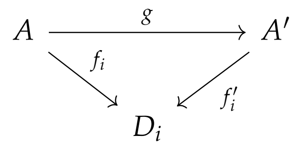
Now we are ready to introduce a (co)limit, one of the fundamental notions in category theory. We start with the category of cones over a diagram.

Definition 15. Let **D** be a diagram with vertices $\{D_i\}$. A cone (over **D**) is a collection $\{f_i : A \to D_i\}$ of arrows such that for any $d : D_i \to D_j$ the diagram



commutes.

An arrow from $\{f_i : A \to D_i\}$ to $\{f'_i : A' \to D_i\}$ is defined by an arrow $g : A \to A'$ such that

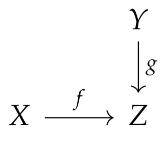


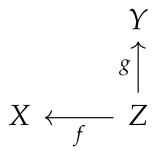
commutes; in such case we say $\{f_i : A \to D_i\}$ factors through $\{f'_i : A' \to D_i\}$.

Definition 16. We call a cone the limit $\lim(\mathbf{D})$ for a diagram **D** if every cone factors uniquely through $\lim(\mathbf{D})$ in $\mathcal{C}$. We define a colimit $\operatorname{colim}(\mathbf{D})$ dually, i.e. by reversing all the arrows in $\mathcal{C}$.

Example 17. The notion of (co)limit is general enough to encompass objects such as:

- initial and terminal object (take an empty diagram)
- product and coproduct (take a pair of objects without arrows)
- pushforward and pullback; these are (co)limits of





Remark 44. Observe that in **Set**, maps of the form $\{*\} \to X$ may be identified with the elements of X. Category theory generalizes this notion to any category **C** and defines

- global elements (global sections) of an object A to be maps $1 \to A$, where 1 is terminal in **C** (if it exists); we use the notation $\Gamma A = \mathrm{Hom}(1, A)$,
- local elements (local sections) of an object A to be maps $B \to A$ for any object B not isomorphic to 1.

To see how category theory enables one to generalize over a well-known structure, first consider the category **Frm** of frames and frame homomorphisms, with objects (frames) defined as complete Heyting algebras (see Definition 9) and morphisms defined as functions preserving finite meets and arbitrary joints. Then category **Loc** of locales is the category opposite to **Frm**. Let X be a topological space with its topology $\mathcal{O}(X)$. It is easy to see that $\mathcal{O}(X)$ is a complete Heyting algebra, and every continuous function $f : X \to Y$ between topological spaces induces a frame map $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. Thus, by defining $\mathcal{O}(f) = f^{-1}$ to be a morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ of locales, we obtain in fact a covariant functor $\mathcal{O}(-) : \mathbf{Top} \to \mathbf{Loc}$ (abusing the notation, a locale corresponding to the frame $\mathcal{O}(X)$ is frequently denoted by X, while locale morphism is usually written as $f : X \to Y$, although not all locales X are of the form $\mathcal{O}(X)$, actually). Moreover, every frame becomes naturally a Lindenbaum–Tarski algebra (i.e. a set of formulae modulo provable equivalence, partially ordered by entailment) for the so-called *geometric* propositional logic, which is essentially an intuitionistic logic with only finite conjunctions (what exactly agrees with the operations frame homomorphisms preserve).

Definition 17. A topos is a category that contains

- a terminal object,
- all pullbacks,
- all exponentials, i.e.
- a subobject classifier, i.e. there exists an object Ω with an arrow $1 \to \Omega$ such that for every monic $A \rightarrowtail B$ there is a unique $\chi_A : B \to \Omega$ such that

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ B & \xrightarrow{\chi_A} & \Omega \end{array}$$

  is a pullback square. In particular, it means that for each subobject A of B there exists a unique "characteristic arrow" $\chi_A : B \to \Omega$.

Example 18. The most natural example of a topos is the **Set** category of sets and functions. Specifically:

- every singleton $\{*\}$ is terminal, since for all sets X there is a unique map $X \to \{*\}$,
- for all sets X, Y, the exponential Y^X is the set of all maps $X \to Y$,
- given maps $f : X \to Z$ and $g : Y \to Z$, the pullback is given by

$$\{(x, y) \in (X, Y) \ : \ f(x) = g(y)\}.$$

- the subobject classifier is the set $\{0, 1\}$, which is nothing more than a codomain of a characteristic function χ_X for every $X \in \mathrm{Ob}(\mathbf{Set})$.

Now, we claim that an internal logic of any topos is the intuitionistic one, due to

Proposition 5. Let $\mathbf{T}$ be a topos and $\mathrm{Sub}(A)$ be a collection of subobjects of $A \in \mathrm{Ob}(\mathbf{T})$ and Ω be a subobject classifier of $\mathbf{T}$. Then $\Gamma\Omega = \mathrm{Hom}(1, \Omega)$ as well as $\mathrm{Sub}(A)$ form Heyting algebras (see A).

The most important type of a topos that we consider in the thesis is that of a (pre)sheaf topos, i.e. the topos of sets "varying" over some base category **C**.

Example 19. For any category **C**, the presheaf category $\mathrm{Set}^{\mathbf{C}^{op}}$ of contravariant functors $\mathbf{C} \to \mathrm{Set}$ is a topos. In particular, in Section **??**, the category **C** becomes the category of C^*- or von Neumann algebras. If **C** is a posetal category, we call $\mathrm{Set}^{\mathbf{C}^{op}}$ a Kripke topos.

Definition 18. A natural numbers object in a topos **T** with terminal object 1 is an object $\mathbb{N}$ with two morphisms:

- $s : \mathbb{N} \to \mathbb{N}$ (successor),
- $0 : 1 \to \mathbb{N}$ (zero)

such that for every diagram $1 \xrightarrow{f} X \xrightarrow{g} X$, there is a unique $u : \mathbb{N} \to X$ such that the following commutes:

$$\begin{array}{ccccc} 1 & \xrightarrow{0} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \searrow^{f} & \downarrow{u} & & \downarrow{u} \\ & & X & \xrightarrow{g} & X \end{array}$$

B.2 Model theory

In the current section we give a very brief overview of model theory that appears throughout the thesis. Although it is not meant to be exhaustive, it should give at least the rudimentary view on the subject and give a sound basis for discussion in Chapters 2, 3 and 6. We start with some basic definitions and observations.

Definition 19. A relational system is a set

$$\mathcal{A} = (A, \mathcal{R}, \mathcal{F}, \mathcal{C})$$

where A is called a universe and families $\mathcal{R} = \{r_i^A\}$, $\mathcal{F} = \{f_i^A\}$, $\mathcal{C} = \{c_i^A\}$ consist of relations on A, functions on A and constants, respectively.

Example 20. Any Boolean algebra (see Appendix A) is a relational system with two binary relations sup, inf, one unary relation $(\cdot)'$ and two constants $0, 1$.

Now, to express the relations between elements of a universe we introduce the notion of a first-order language.

Definition 20. The language is a set

$$\mathcal{L} = (R, F, C, X, S)$$

where the families $R = \{r_i\}$, $F = \{f_i\}$, $C = \{c_i\}$ consist of relational, function symbols and constants respectively, $X = \{x_n\}$ is a set of variables and $S = \{=, \neg, \implies, \forall\}$ is a set of logical symbols.

Remark 45. Observe that the structure of a language lets one to interpret the elements of R, F, C as the elements of $\mathcal{R}, \mathcal{F}, \mathcal{C}$ in a particular system $\mathcal{A}$, the variables $\{x_n\}$ as elements of A and the elements of S as logical operators. Clearly, one obtains the other logical operators by rules such as

$$(\phi \vee \psi) \equiv (\neg\phi \implies \psi)$$

With the help of X, C and F one builds recursively more complicated expressions, called terms, e.g. $f_j(x_i, c_k)$. In general

$$\mathrm{Tm}_0 = X \cup C, \quad \mathrm{Tm}_{l+1} = \mathrm{Tm}_l \cup \{f_j(t_1, \dots t_m) \, : t_1, \dots, t_m \in \mathrm{Tm}_l\}$$

Finally the set of all terms is equal to $\mathrm{Tm} = \bigcup \mathrm{Tm}_l$. We define free variables $V(t)$ of any term t may be defined by

$$V(x_n) = \{x_n\}, \, V(c) = 0 \text{ for } x_n \in X, c \in C,$$
$$V\left(f(t_1, \dots, t_m)\right) = V(t_1) \cup \dots \cup V(t_m)$$

Eventually, we build longer expressions called formulae out of terms and logical symbols as follows:

$$\begin{aligned} \mathrm{Fm}_0 &= \{t = s \, : \, t, s \in \mathrm{Tm}\} \cup \{r_i(t_1, \dots, t_m) \, : \, t_1, \dots, t_m \in \mathrm{Tm}\}, \\ \mathrm{Fm}_{l+1} &= \mathrm{Fm}_{l+1} \cup \{\neg\phi \, : \, \phi \in \mathrm{Fm}_l\} \cup \{\phi \implies \psi \, : \, \phi, \psi \in \mathrm{Fm}_l\} \cup \\ &\qquad \cup \{\forall x_n \, (\phi) \, : \, \phi \in \mathrm{Fm}_l\} \end{aligned}$$

Thus the set of all formulae comes as $\mathrm{Fm} = \bigcup \mathrm{Fm}_l$ and they extend the most fundamental statements involving equality and relations with the help of logical symbols. Formulae' free variables are defined recursively as

$$\begin{aligned} V(t = s) = V(t) \cup V(s), \quad V\left(r_i(t_1, \dots, t_m)\right) &= V(t_1) \cup \dots (t_m), \\ V(\neg\psi) = V(\phi), \quad V(\phi \implies \psi) &= V(\phi) \cup V(\psi), \\ V\left(\forall x_n \, (\phi)\right) &= V(\phi) \setminus \{x_n\} \end{aligned}$$

We call ϕ a sentence if $V(\phi) = \varnothing$. Since we would like to interpret the language in a given system, we introduce an interpretation function $i : \mathrm{Tm} \supseteq Y \to A$ by

$$i(c_k) = c_k^A, \quad i\left(f_j(t_1, \dots, t_m)\right) = f_j^A\left(i(t_1, \dots, i(t_m)\right)$$

The following simple example show how an interpretation gives a truth valuation of formulae.

Example 21. Let $\phi(x) = "x = 0"$; we have $V(\phi) = \{x\}$. Let $i, j : x \mapsto \mathbb{R}$ be two interpretations such that $i(x) = 0$ and $j(x) = 1$. Clearly i makes $\phi(x)$ true, while j makes $\phi(x)$ false.

We now make the statement "ϕ is true in a relational system $\mathcal{A}$" (written $\mathcal{A} \models \phi$) precise; we define inductively

$$\begin{aligned} \left(\mathcal{A} \models (t = s)[i]\right) &\equiv \left(i(t) = i(s)\right) \\ \left(\mathcal{A} \models r_i(t_1 \dots, t_m)[i]\right) &\equiv r_i\left(i(t_1), \dots, i(t_m)\right) \\ \left(\mathcal{A} \models \neg\phi[i]\right) &\equiv \left(\text{not } \mathcal{A} \models i(\phi)\right) \\ \left(\mathcal{A} \models (\phi \implies \psi)[i]\right) &\equiv \left(\text{not } \mathcal{A} \models i(\phi) \text{ or } \mathcal{A} \models i(\psi)\right) \\ \left(\mathcal{A} \models \forall x_n \, (\phi) \, [i]\right) &\equiv \left(\text{for every } a \in A, \, \mathcal{A} \models i_x^a(\phi)\right) \end{aligned}$$

where i_x^a denotes x_n is substituted with a under i.

We are ready now to introduce the notion of a model of a theory.

Definition 21. A system $\mathcal{A}$ is called a model of a set T of sentences (a theory T) in a language $\mathcal{L}$, if $\mathcal{A} \models \phi$ for every $\phi \in T$.

Example 22. We refer to the following examples from everyday mathematics.

- To find an example of a group, such as $(\mathbb{Z}, +)$ (e.g. nonsingular matrices with matrix multiplication), what one provides is essentially a model for a certain theory (i.e. the set of axioms that we all get to know during undergraduate linear algebra course). In other words, one verifies that $(\mathbb{Z}, +)$ contains the associative binary operation together with the identity and inverse elements with the respective properties. Moreover, note that both abelian and nonabelian groups exist, thus (non)commutativity property cannot be a logical consequence of group axioms (we say such property is independent of the axioms). In other words, one may include the (non)commutativity in the group axioms without a contradiction.

- Another example is the category **Set** of sets and functions (see Section B.1 is a model of ZFC axioms.

By $\mathrm{Mod}(T)$ we denote the class of all models of the set of sentences T. Then, a class of systems $\mathcal{K}$ is said to be axiomatizable, if $\mathcal{K} = \mathrm{Mod}(T)$ for some T; in this case we call T an axiomatics for $\mathcal{K}$. A theory $\mathrm{Th}(\mathcal{K})$ of a class $\mathcal{K}$ is defined by

$$\mathrm{Th}(\mathcal{K}) = \{\phi : \forall \mathcal{A} \in \mathcal{K} \ (\mathcal{A} \models \phi)\}$$

Clearly, if $\mathcal{K}$ is axiomatizable, then $\mathcal{K} = \mathrm{Mod}\ (\mathrm{Th}(\mathcal{K})$.

Example 23. The category **Grp** of groups and group homomorphisms (see Section B.1) is (first-order) axiomalizable, as **Grp** $= \mathrm{Mod}(T)$ for the set T of group axioms. However, some of the well-known structures are not first-order axiomatizable; in particular, for the category **Top** of topological spaces and continuous functions there is no first-order theory T such that **Top** $= \mathrm{Mod}(T)$; similar case holds for well-ordered sets.

In order to make the results of Gödel comprehensible, let us consider another, fundamental example — the Peano arithmetic.

Example 24. Let $\mathcal{N} = (N, +, \cdot, 0, 1)$ where N is the set of natural numbers, the operations $+, \cdot$ are addition and multiplication respectively, and $0, 1$ are constants. The axioms of Peano, called PA, consist of

$$\begin{aligned} a+b=b+a, \quad a\cdot b=b\cdot a \\ (a+b)+c=a+(b+c), \quad (a\cdot b)\cdot c=a\cdot(b\cdot c) \\ (a+b)\cdot c=(a\cdot c)+(b\cdot c) \\ a+\Delta_0=a, \quad a\cdot\Delta_1=a \\ a+c=b+c \implies a=b \\ a\neq\Delta_0 \text{ iff } \exists b\ (a=b+\Delta_1) \\ \forall\phi\,\forall a\left(\phi(\Delta_0/a)\wedge\forall a\left(\phi \implies \phi\left((a+\Delta_1)/a\right)\right) \implies \forall a\,(\phi)\right) \end{aligned}$$

In general, one notices that axioms are not the only sentences that we find true in a given model. Namely, there exist theorems proved by the axioms and some predefined inference rules, which are satisfied under any interpretation in any relational system. These inference rules are usually called the axioms of logic; let us denote them by LOG. Observe that particular LOG gives rise to what is called deductive system.

Example 25. In this work we consider mostly two types of deductive systems: classical and intuitionistic first-order logic. It is not really necessary to give a full treatment on the rules of classical deduction, as it is a part of usual reasoning in mathematics; however, it is inevitable to note the property that distinguishes it from intuitionistic logic, namely availability of the inference

$$\frac{\neg\neg\phi}{\phi}$$

Equivalently we might say that the truth of $\phi \vee \neg\phi$, called the law of excluded middle, is not derivable in intuitionistic logic.

Definition 22. We define theorems of a theory T inductively, starting with T itself, together with the axioms of logic:

$$T_0 = T \cup \mathrm{LOG}, \tag{B.1}$$

$$T_{n+1} = T_n \cup \{\psi : \exists\phi \ (\psi \in T_n \text{ and } \phi \implies \psi \in T_n)\}. \tag{B.2}$$

The elements of a set

$$T^* = \bigcup_{n\in\mathbb{N}} T_n$$

are called theorems of a theory T. We write $T \vdash \phi$ iff $\mathcal{A} \models \phi$ in every $\mathcal{A} \in \mathrm{Mod}(T)$. If $T = \varnothing$, we write $\vdash \phi$ for $T \vdash \phi$, what makes ϕ a tautology, i.e. ϕ can be proved on the basis of LOG.

Definition 23. A theory T is inconsistent if both $T \vdash \phi$ and $T \vdash \neg\phi$ for some ϕ; otherwise we say T is consistent, written $\mathrm{Con}(T)$.

If a theory T is inconsistent, T is usually depicted as nonsensical, since one may prove that $T \vdash \psi$ and $\mathcal{A} \models \psi$ for any ψ and $\mathcal{A} \in \mathrm{Mod}(T)$. On the other hand, appending theorems to a consistent theory T leaves T consistent, i.e. $\{\phi : T \vdash \phi\}$ is consistent.

The following notion of decidability will be the cornerstone of the Continuum Hypothesis that we discuss in Chapter 2.

Definition 24. A formula ϕ is called decidable in T if either $T \vdash \phi$ or $T \vdash \neg\phi$. Otherwise ϕ is independent of T, equivalently $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ (written $T + \phi$ and $T + \neg\phi$, respectively) are consistent. We call T a complete theory if there are no formulae independent of T; otherwise T is called incomplete.

Example 26. Let us refer again to the first-order theory of groups; for a provable sentence, consider the uniqueness of neutral element in any group. At the same time, one cannot prove from group axioms that the group multiplication is commutative; this may be true in some models (i.e. in groups that are abelian such as $(\mathbb{Z}, +)$) and false in the other (e.g. $\mathrm{GL}_n(\mathbb{R})$).

The following theorem formalizes the notion of a proof.

Theorem 37. Consider a theory T and suppose that $T \vdash \phi$; then there exists a finite sequence of formulae $(\phi_0, \ldots, \phi_n)$ such that $\phi_n = \phi$ and for every $i \leq n$ either $\phi_i \in T$ or $\phi_i \in$ LOG or ϕ_i arises by modus ponens from ϕ_{i-2} and ϕ_{i-1} (i.e. from ϕ_{i-2} and $\phi_{i-2} \implies \phi_{i-1}$ one concludes ϕ_{i-1}). We call such sequences $(\phi_0, \ldots, \phi_n)$ proofs.

Definition 25. Let M be a class and ϕ be a formula. We define the relativization ϕ^M of ϕ with respect to M inductively by

$$\begin{array}{rcl}
(x = y)^M & \text{is} & x = y \\
(x \in y)^M & \text{is} & x \in y \\
(\phi \wedge \psi)^M & \text{is} & \phi^M \wedge \psi^M \\
(\neg\phi)^M & \text{is} & \neg\phi^M \\
(\exists x\,(\phi))^M & \text{is} & \exists x \left(x \in M \wedge \phi^M\right)
\end{array}$$

for $x, y \in M$. Furthermore, ϕ is true in M i.e. $M \models \phi$, if ϕ^M holds. Consequently, ϕ is called M-absolute if ϕ^M holds iff ϕ does, i.e. $V \models \phi$, and a set is called M-absolute if it is defined by an M-absolute formula. Finally, a formula (a set) is called absolute if it is M-absolute for every standard transitive ZFC model M.

Example 27. Formulae such as "x is a real number", "x is an ordinal" (see B.4) or "$x \in y$" are absolute, while "x is a cardinal" or "x is the set of real numbers" are not.

We state two fundamental results due to Gödel, namely his completeness and incompleteness theorems.

Theorem 38. (Gödel's completeness theorem) Every consistent set of sentences has a model.

Remark 46. Actually, there is another, equivalent version of completeness: if $M \models \phi$ for every model M of T, then $T \vdash \phi$ (the converse also holds and is known as the soundness).

Theorem 39. (Gödel's incompleteness theorem) Suppose a theory T includes PA, i.e. there exists a formula $M(x)$ defining a class M such that $T \vdash \exists x\,(M(x))$ and $\mathrm{PA}^M = \{\phi^M : \phi \in \mathrm{PA}\} \subseteq T$, i.e. T contains all the axioms of PA, relativized to $M(x)$. Then T cannot be both consistent and complete. Moreover $T \nvdash \mathrm{Con}(T)$, i.e. T cannot prove its own consistency.

Remark 47. It holds that ZFC includes PA, therefore the sentence "ZFC is consistent" is not provable in ZFC, provided ZFC is consistent (what we assume implicitly throughout the thesis to ensure there exists a model of ZFC due to Theorem 38).

The following theorem proves that cardinality of a model is not an invariant.

Theorem 40. (Löwenheim–Skolem) If a set of sentences T written in the language $\mathcal{L}$ has an infinite model, then it has a model of arbitrary cardinality $\geq |\mathcal{L}|$.

Since PA (see Example 24) has an infinite, countable model — standard natural numbers $\mathbb{N}$ — it has also a model of any infinite cardinality by Theorem 40. Such uncountable models of PA are called non-standard. Let us end this section with an important remark considering models of second-order theories.

Remark 48. Mathematics outside set theory, such as calculus, linear algebra, geometry etc., usually considers the objects of $\mathbb{N}$ and $\mathbb{R}$ as unique entities. But we already observed that, due to Theorem 40, models that are already infinite "generate" models of any cardinality and it applies also to first-order PA. What lets to think about $\mathbb{N}$ and $\mathbb{R}$ as unique, is the second-order theoretic characterization of these structures.

With all the tools introduced above, let us focus on set theory in what follows.

B.3 ZFC theory of sets

For the sake of completeness, let us recall the first-order theory of ZFC [91]:

1. (set existence) $\exists x\,(x = x)$
2. (extensionality) $\forall x\,\forall y\,(\forall(z \in x \iff z \in y) \implies x = y)$
3. (foundation) $\forall x\,\left[\exists y\,(y \in x) \implies \exists y\,(y \in x \wedge \neg\exists z\,(z \in x \wedge z \in y))\right]$
4. (comprehension scheme) for each formula ϕ with free variables among $x, z, w_1, \ldots, w_n$

$$\forall z\,\forall w_1 \ldots \forall w_n\,\exists y\,\forall x\,(x \in y \iff x \in z \wedge \phi)$$

5. (pairing) $\forall x\, \forall y\, \exists z (x \in z \wedge y \in z)$
6. (union) $\forall \mathcal{F}\, \exists A\, \forall Y\, \forall x\, (x \in Y \wedge Y \in \mathcal{F} \implies x \in A)$
7. (replacement scheme) for each formula ϕ with free variables among $x, y, A, w_1, \ldots, w_n$

$$\forall A\, \forall w_1 \ldots \forall w_n\, \left[\forall x \in A\, \exists! y\, \phi \implies \exists Y\, \forall x \in A\, \exists y \in Y\, \phi\right]$$

8. (infinity) $\exists x\, (\varnothing \in x \wedge \forall y \in x\, \left(y \cup \{y\} \in x\right))$
9. (power set) $\forall x\, \exists y\, \forall z\, (z \subset x \implies z \in y)$
10. (choice) $\forall A\, \exists R\, (R \text{ well-orders } A)$

Notice that the language of set theory is countable and it has an infinite model. Consequently, there exists a countable model for ZFC as indicated by Theorem 40. This phenomenon might initially appear paradoxical (and is sometimes referred to as Skolem's paradox). This paradox arises because every ZFC model contains some uncountable objects, such as $\mathbb{R}$. This is due to the fact that cardinality is not an absolute concept in general. We will see that e.g. $\mathbb{R}^M$ is uncountable "inside M" but merely countable when viewed "from the outside".

B.4 Ordinals and cardinals

To understand forcing better, it is inevitable to be familiar with the concepts of ordinal numbers a set cardinality. Recall that we call a set x transitive whenever $z \in y$ and $y \in x$ implies $z \in x$, i.e. the membership is transitive on x.

Definition 26. An ordinal α is a transitive set that is well-ordered by $\in$.

Example 28. Consider the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$; let $0 = \varnothing$ and for each $n \in \mathbb{N}$ define the successor $S(n) = n \cup \{n\}$, e.g.

$$1 = S(0) = \{\varnothing\} = \{0\}, \quad 2 = S(1) = 1 \cup \{1\} = \{\varnothing\} \cup \{\{\varnothing\}\} = \{0, 1\} \text{ etc.} \tag{B.3}$$

In fact, asserting that the natural numbers are the smallest set that satisfies the axiom of infinity provides one way to define $\mathbb{N}$. Thus we can see that every natural number $n = \{0, 1, 2 \ldots, n-1\} \in \mathbb{N}$ is an ordinal.

Let us denote the proper class of all ordinals by **On**. It is easy to show that **On** is well-ordered by $\in$, and therefore for $\alpha, \beta \in \mathbf{On}$ we write $\alpha \leq \beta$ if $\alpha \in \beta$ or $\alpha = \beta$. Furthermore, defining the sum $\alpha + \beta$ to be the unique ordinal that is isomorphic to the well-ordered concatenation of α and β, we see that the operation (B.3) extends to infinite ordinals as well. Again for each $\alpha \in \mathbf{On}$, applying the successor function $S(\alpha)$ splits **On** into two disjoint collections: successors, i.e. ordinals α for each there exists β such that $\alpha = S(\beta)$, and limits, i.e. non-zero ordinals that are not successors. Obviously, natural numbers (i.e. finite ordinals) β satisfy $\forall \alpha \leq \beta\, (\alpha = 0 \vee \alpha \text{ is a successor})$. Following our intuition, we characterize ω to be the least limit ordinal (or the first infinite ordinal), which shows that $\omega \equiv \mathbb{N}$. Since nothing prevents us from performing $\omega + 1 = S(\omega)$ or even replacing each element of ω with a copy of ω, concatenate the result and obtain an ordinal called ω^2, continuing this procedure leads to ω^3, ω^4 up to ω^ω (the last results from concatenating $\omega, \omega^2, \omega^3, \ldots$). Note that all of these are countable. On the other hand we have [172]

Remark 49. Every set is bijective to some ordinal, thus there exists an uncountable ordinal.

Based on the above, we introduce the notion of a cardinal, which is an ordinal that is not bijective with any preceding ordinal. Thus the finite cardinals correspond to elements of ω and we adopt the notation $\aleph_n$ for consecutive infinite cardinals, starting with $\aleph_0 = \omega$. Accordingly, for any set X, the smallest ordinal bijective with X is a cardinal called the cardinality of X and denoted by $\text{card}(X)$.

B.5 Forcing and Boolean-valued models

To emphasize the historical importance of the forcing technique [47], let us remind that the method was specifically created to address the long-standing continuum hypothesis (CH), stated as the following conjecture: *There is no S such that $\mathbb{N} \subsetneq S \subsetneq \mathbb{R}$ and there are no surjections $\mathbb{N} \twoheadrightarrow S$ and $S \twoheadrightarrow \mathbb{R}$.* Thus, informally speaking, CH states that there are no sets of cardinality strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$ or, in other words, all infinite subsets of $\mathbb{R}$ are of the size either $\mathbb{N}$ or $\mathbb{R}$. Again, refering to Section B.4, it means that continuum is the next cardinal after $\aleph_0$, i.e. $\aleph_1 = 2^{\aleph_0}$.

Remark 50. Prior to the invention of forcing, it had been already known that CH is consistent with ZF due to Gödel [75]. It was achieved by constructing a particular model, called (Gödel's) constructible universe built as follows [90]. We call a set X to be definable over a model M if there exists a ZF-formula ϕ and $a_1, \ldots, a_n \in M$ such that

$$X = \{x \in M \,:\, M \models \phi(x, a_1, \ldots, a_n)\}. \tag{B.4}$$

A class of definable sets $\text{def}(M)$ is therefore equal to

$$\text{def}(M) = \{X \subset M \,:\, X \text{ is definable over } M\}.$$

Then we define recursively:

- $L_0 = \varnothing$, $L_{\alpha+1} = \text{def}(L_\alpha)$
- $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ if α is a limit ordinal
- $L = \bigcup_{\alpha\in\text{On}} L_\alpha$

Finally one shows that L is a model of ZFC + CH. The very first notion that forcing relies on is a partial order P called a forcing notion, with the elements referred to as forcing conditions. We call $p, q \in P$ compatible if there is $r \in P$ such that $r \le p$, $r \le q$ (p, q have common lower bound), and write $p \perp q$ is p, q are not compatible. Thus, under the interpretation of P as containing pieces of partial information, compatibility of p, q means these are mutually consistent, i.e. they can be consistently refined. Moreover, we will always demand P to be separative (also called refined), i.e. if $p \not\le q$ implies $\exists r (r < q \text{ and } r \perp p)$. Again, this means that whenever p is not a refinement of q, there exists a condition r refining q that is incompatible with p.

Example 29. Consider a function $F : \mathbb{N} \to \{0, 1\}$. One can provide a partial information about f by handing finite partial functions $f : X \subset \mathbb{N} \to \{0, 1\}$ that agree with F on X. Observe that $g \le f$ if $f \subseteq g$ (g extends f) partially orders these functions, and $f \perp g$ if f, g do not agree on their common domain. Finally, if f does not extend g, there always exists an extension of g that does not agree with f, thus above inclusion gives a separative poset.

One defines a P-name to be the set

$$M \ni n = \{(m, p) \,:\, m \text{ is a } P-\text{name},\ p \in P\}.$$

Names are interpreted as

$$n^G = \{m^G : (m,p) \in n, \quad p \in G\}$$

and finally $M[G] = \{n^G : n \in M \text{ is a } P\text{-name}\}$. Now we will show how the above construction can be rewritten by replacing posets with Boolean algebras. It is good to start with a construction of a two-valued universe of sets by a transfinite recursion; the same set of procedures will serve to define a more general Boolean-valued model (recall that $\{0,1\}$ is the simplest non-trivial Boolean algebra). First of all, the successor operation (B.3) can be generalized; it is usually attributed to von Neumann that the standard universe of sets can be built similarly to Remark 50:

$$\begin{aligned} V_0 &= \varnothing, \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha), \\ V &= \bigcup_{\alpha \in \mathbf{On}} V_\alpha \end{aligned} \tag{B.5}$$

Since the subsets may be identified with characteristic functions, we may stress the two-valuedness of standard sets by defining for each ordinal α

$$\begin{aligned} V_0^{(2)} &= \varnothing, \\ V_\alpha^{(2)} &= \{x : x \text{ is a function } \wedge \operatorname{ran}(x) \subseteq 2 \wedge \exists \xi < \alpha \left(\operatorname{dom}(x) \subseteq V_\xi^{(2)}\right)\}, \\ V^{(2)} &= \{x : \exists \alpha \left(x \in V_\alpha^{(2)}\right)\} \end{aligned} \tag{B.6}$$

One can show that $V^{(2)}$ is isomorphic to the standard universe of sets V [20].

Let B be a complete Boolean algebra (see Appendix A). We define the universe $V^{(B)}$ of B-valued sets similarly to (B.6), by replacing the Boolean algebra $2 = \{0,1\}$ by B. Hence, B-valued sets can be thought of as characteristic functions with a more general codomain B. Now, to every sentence ϕ we assign a truth value $[\![\phi]\!] \in B$, just like all sentences are either true or false in the standard universe V. In particular, we say that a sentence is true (or holds with a probability 1), denoted by $V^{(B)} \models \phi$, if $[\![\phi]\!] = 1$. Then we have [20]

Theorem 41. All the axioms and theorems of ZFC are true in $V^{(B)}$ (we say that $V^{(B)}$ is a B-valued ZFC model).

Remark 51. In the same way one defines a Boolean-valued model M^B where M is a countable transitive ZFC model; here we demand that $B \in M$ is M-complete (i.e. it is complete inside M), all the B-valued functions are elements of M and the recursion is performed over ordinals in M. Finally, M^B is a B-valued ZFC model as well.

Remark 52. Despite the fact that both V^B and M^B satisfy ZFC axioms, neither V^B nor M^B are not ZFC models as long as B is nontrivial, since the truth values of formulae can be neither true nor false in general. (Recall that if N is a ZFC model, then for every formula either $N \models \phi$ or $N \models \neg\phi$.)

Observe that the approaches are equivalent due to [20]

Proposition 6. The poset P is separative iff it is order isomorphic to a dense subset of a complete Boolean algebra. We call such P a basis for B.

Theorem 42. Let B be a complete Boolean algebra and the poset P be a basis for B. For every B-sentence ϕ and $p \in P$ we have

$$p \Vdash \phi \text{ iff } p \leq [\![\phi]\!]^B.$$

Suppose that U is an ultrafilter in B (see Definition 8) and define an equivalence relation on M^B

$$x \sim_U y \iff [\![x = y]\!] \in U$$

with the equivalence classes denoted by $x^U = \{y \in M^B : x \sim_U y\}$. Further we define the relation $\in_U$

$$x^U \in_U y^U \iff [\![x \in y]\!] \in U$$

and finally we quotient Boolean-valued model by the ultrafilter as

$$M^B/U = \left(\{x^U : x \in M^B\}, \in_U\right).$$

Above quotient is already a ZFC model due to

Theorem 43. For any formula ϕ and $x_1, \dots, x_n \in M^B$ it holds

$$M^B/U \models \phi(x_1^U, \dots, x_n^U) \iff [\![\phi(x_1, \dots, x_n)]\!] \in U. \tag{B.7}$$

Since $1 \in U$ by definition of an ultrafilter, all sentences true in M^B remain true in M^B/U. In particular, M^B/U is a ZFC model by (B.7) and by Theorem 41. In order to ensure M^B/U to be standard (i.e. $\in_U$ is well-founded) and, even more importantly, to have $U \in M^B/U$, one requires U to be M-generic: this means that U as a subset of a poset $P = B \setminus \{0\}$ intersects all dense subsets of P such that $D \in M$. Furthermore if U is M-generic, then the Mostowski's collapse $M[U]$ of M^B/U is the smallest standard transitive ZFC model such that both $M \subseteq M^B/U$ and $U \in M^B/U$.

The following lemma is a crucial condition emphasizing the importance of atomless Boolean algebras in the forcing technique [90].

Lemma 21. Let B be a Boolean algebra in M and suppose there exists a generic ultrafilter U on B. Then, B is atomless iff $U \notin M$.

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