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FUNCTIONAL INEQUALITIES CONNECTED TO SUGENO INTEGRAL AND ITS APPLICATIONS

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Contents

Abstract

This dissertation presents functional inequalities connected to Sugeno integral and its applications. We focus on the Hermite-Hadmard inquality. So we start with the computer approach to solve the general form of Hermite-Hadamard inequality and to the best of our knowledge, this is the first work where a computer program may be used to solve functional inequalities.
We then study the extension of Hermite-Hadamard inequality for the case of quasi-arithmetically convex functions and it's Sugeno intergal counter part which provides a generalization and it acts as a generator for other means, in particular linear, harmonic, geometric among others and this is followed by the study of the Lagrangian mean (non-arithmetic mean) which leads to the characterization of the logarithmic mean.
Then, on upper Hermite–Hadamard inequalities for geometric-convex and log-convex functions. This is a correction on a result by J. Sándor which is contained in article [55] where author claims, among others, that theorem 6.1 holds (cf. Theorem 2.5 in [55]).
Finally, we present the applications of fuzzy measure theory where we first propose an iterative approach to obtain the optimal value for λ without having to solve complex polynomial functions. And then application of non-additive fuzzy measures as an alternative to traditional risk metrics like standard deviation. So we consider a Markovitz-like portfolio selection problem, where we use a fuzzy measure (a transfomation of Sugeno lambda-measure) and a d-Choquet integral to form efficient frontier. Due to the limitations of Modern Portfolio Theory (MPT) and its reliance on normal distribution assumptions, we introduce non-additive fuzzy measure, which do not assume specific probability distributions. This approach accommodates imprecision and uncertainty in financial markets, providing a more comprehensive understanding of portfolio risk. By considering diversification and asset characteristic dependencies, non-additive fuzzy measures offer a promising avenue for more accurate risk analysis and informed investment decisions.

Keywords: Functional equations, Hermite–Hadamard inequalities, Fuzzy measure, Sugeno integral, convex(concave) functions, Risk management, Modern Portfolio Theory, stochastic orderings, Computer assisted methods, Python.

Streszczenie

Niniejsza rozprawa przedstawia nierówności funkcyjne związane z całką Sugeno i jej zastosowaniami. Skupiamy się na nierówności Hermite'a-Hadmarda. Wprowadzamy podejście komputerowe do rozwiązania ogólnej postaci nierówności Hermite'a-Hadamarda; wedle naszej najlepszej wiedzy jest to pierwsza praca, w której można zastosować program komputerowy do rozwiązania nierówności funkcyjnych.

Następnie badamy rozszerzenie nierówności Hermite'a-Hadamarda w przypadku funkcji quasi-arytmetycznie wypukłych, w szczególności dla generatorów liniowych, harmonicznych, geometrycznych. Badamy też średnie lagrange'owskie, co prowadzi do charakterystyki średniej logarytmicznej.

Następnie przedstawiono notatkę do wyniku J. Sándora, która stanowi korektę wyniku zawartego w artykule [40], gdzie autor twierdzi m.in., że twierdzenie 6.1 jest prawdziwe (por. Twierdzenie 2.5 w [55]).

Na koniec przedstawiamy zastosowania teorii miary rozmytej. Najpierw proponujemy podejście iteracyjne w celu uzyskania optymalnej wartości λ bez konieczności rozwiązywania złożonych funkcji wielomianowych. Następne zastosowanie dotyczy miary rozmytej w zarządzaniu ryzykiem portfela, gdzie proponujemy nową, nieaddytywną (rozmytą) funkcję agregującą, która nie tylko nie zakłada żadnego rozkładu, ale oddaje dywersyfikację i zależność w charakterystyce aktywów.

Słowa kluczowe: Równania funkcyjne, nierówności Hermite'a - Hadamarda, miary rozmyte, całka Sugeno, funkcje wypukłe (wklęsłe), zarządzanie ryzykiem, nowoczesna teoria portfelu, porządki stochastyczne, metody komputerowe rozwiązywania równań i nierówności funkcyjnych, język programowania Python

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1 Introduction

Measure is one of the most important concepts in mathematics, for example the concept of integration with respect to a given measure. In the classical definition of measure we use additivity property. Additivity is very effective in many applications, but in many real world problems we do not require measure with respect to the additivity feature, for example in fuzzy logic, artificial intelligence, data mining, decision making theory among others additivity would be redundant. So the fuzzy measure doesn't require additivity in most cases, in fuzzy measure we require monotonicity related to inclusion of sets.The development of fuzzy measure theory has been motivated by the increasing apprehensiveness that the additivity property of classical measures is in some applications context too restrictive and consequently unrealistic.
In mathematics, fuzzy measure theory considers generalized measures in which the additive property is replaced by the weaker property of monotonicity. The central concept of fuzzy measure theory is the fuzzy measure which was introduced by Choquet in 1953 and independently defined by Sugeno in 1974 in the context of fuzzy integrals. Sugeno integral is one of the most important fuzzy integrals, which has many applications in various fields.
The thesis presents the inequalities connected to Sugeno integral in particular the Hermite-Hadamard Inequality. The Hermite-Hadamard inequality is the first fundamental result for convex functions defined on a interval of real numbers with a natural geometrical interpretation and many applications. It has attracted and continues to attract much interest in elementary mathematics. Many mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions as seen in [18] [11], [15],[19], [27], [29], [30], [31], [40] etc.

Definition 1.1. The classical convexity is defined as follows:
A function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is a real interval, is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the inequality is reversed f is said to be concave.

The classical Hermite-Hadamard inequality provides estimates of the mean value of a convex function $f : [a,b] \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

The thesis is composed of 6 chapters. Chapters 1 and 2 are devoted to the general introduction and definitions of key terms.
In chapter 3, we present a computer program for solving inequalities of the form

$$a_1 f(\alpha_1 x + (1-\alpha_1)y) + \cdots + a_n f(\alpha_n x + (1-\alpha_n)y) \leqslant \frac{1}{y-x}\int_x^y f(t)dt \quad (2)$$

where a_i, α_i are non-negative real numbers and $\sum_{i=1}^{n} a_i = 1$ and $f : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous.
Chapter 4 is devoted to the Sugeno Integral of Hermite-Hadamard Inequality for case of quasi-arithmetically convex functions. Here we study the extension of Hermite-Hadamard Inequality for the case of quasi-arithmetically convex functions and it's Sugeno intergal counter part which provides a generalization and it acts as a generator for other means, in particular linear, harmonic, geometric among others. This is followed by the study of the Lagrangian mean (non-arithmetic mean) in Chapter 5 that's the characterization of the logarithmic mean. And then, on upper Hermite–Hadamard inequalities for geometric-convex and log-convex functions. This is a correction on a result by J. Sándor which is contained in article [55] where author claims, among others, that theorem 6.1 holds (cf. Theorem 2.5 in [55]).
Finally, we present the applications of fuzzy measure theory where we first propose an iterative approach to obtain the optimal value for λ without having to solve complex polynomial functions.This iterative method proves advantageous, particularly in real-world scenarios characterized by larger domains, surpassing the limitations of conventional techniques and that's the gradient descent. Next is the application of non-additive fuzzy measures as an alternative to traditional risk metrics like standard deviation. So we consider a Markovitz-like portfolio selection problem, where we use a fuzzy measure (a transfomation of Sugeno lambda-measure) and a d-Choquet integral to form efficient frontier. Due to the limitations of Modern Portfolio Theory (MPT) and its reliance on normal distribution assumptions, we introduce non-additive fuzzy measure, which do not assume specific probability distributions. This approach accommodates imprecision and uncertainty in financial markets, providing a more comprehensive understanding of portfolio risk. By considering diversification and asset characteristic dependencies, non-additive fuzzy measures offer a promising avenue for more accurate risk analysis and informed investment decisions.

List of publications

Most of the results presented in this dissertation are based on the following articles:

(1) **T. Nadhomi**, Sugeno Integral for Hermite–Hadamard inequality and quasi-arithmetic means. Annales Mathematicae Silesianae 37 (2023), no. 2, 294–305. https://doi.org/ 10.2478/amsil-2023-0007.

(2) C. P. Okeke, W. Ogala and **T. Nadhomi**, On symbolic computation of C.P. Okeke functional equations using Python programming language. Aequat. Math. (2023). https://doi.org/10.1007/s00010-023-01008-2.

(3) J. Sándor, Corrigendum to "On upper Hermite–Hadamard inequalities for geometric-convex and log-convex functions" [Notes on Number Theory and Discrete Mathematics, 2014, Vol. 20, No. 5, 25–30]

(4) **T. Nadhomi**, M. Sablik and J. Sikorska, On a characterization of the logarithmic mean. Submitted.

(5) **T. Nadhomi**, C. P. Okeke, M. Sablik and T. Szostok, On a class of functional inequalities, a computer approach. Submitted.

(6) **T. Nadhomi**, C. P. Okeke and M. Sablik, Portfolio selection based on a fuzzy measure. Submitted.

2 Preliminaries

To begin with, is to have an overview of fuzzy measure theory which will give us the insight of fuzzy measures and fuzzy integral.
Firstly, we introduce the following notation: let $X_i, i \in \{1, \cdots, n\}$, be pairwise disjoint non-empty sets that is $X_i \cap X_j = \varnothing, i \neq j$. We denote by X the union of all X_i sets, namely

$$X = \bigcup_{i=1}^{n} X_i.$$

The set $\mathcal{X}$ consists of all "classes" generated by $X_1, \ldots, X_n$, namely

$$\mathcal{X} = \{\varnothing, X_1, \ldots, X_n, X_1 \cup X_2, \ldots, X_1 \cup X_n, \ldots, X_1 \cup X_2 \cup \cdots \cup X_n = X\}.$$

It is a straightforward matter to check that $\mathcal{X}$ is a σ-algebra of subsets of X. Within this context, we introduce the notation $\sigma(j)$, where j belongs to the set $\{0, 1, \cdots, n\}$. It represents a collection of sets, where each member contains precisely j disjoint sets. In simpler terms, $\sigma(0) = \varnothing$, $\sigma(1) = \{X_1, \cdots, X_n\}$, $\sigma(2) = \{\{X_1 \cup X_2\}, \ldots, \{X_1 \cup X_n\}, \{X_2 \cup X_3\}, \ldots, \{X_2 \cup X_n\}, \ldots, \{X_{n-1} \cup X_n\}\}$, and so on, until $\sigma(n) = \{\{X_1 \cup X_2 \cup \cdots \cup X_n\}\}$.
The sets $X_1, \ldots, X_n$ are the atoms of a finite Boolean algebra which is isomorphic to the powerset of $\{1, \ldots, n\}$. Indeed the family $\mathcal{X}$ is isomorphic to all subsets of $\{1, \ldots, n\}$.

2.1 Fuzzy measure theory

Definition 2.1. (Probability measure)
Adopting the above notation, we say that the function $Pr : \mathcal{X} \to [0, 1]$ is a probability measure on the space $(X, \mathcal{X})$ if and only if Pr satisfies the following requirements:

(1) $Pr(\varnothing) = 0$, $Pr(X) = 1$,

(2) for any $A, B \in \mathcal{X}$ with $A \cap B = \varnothing$, the following equality holds

$$Pr(A \cup B) = Pr(A) + Pr(B).$$

When addressing uncertainty, the conventional approach has been to employ additive measures such as the probability measure (Pr). However, the advent of fuzzy sets has revealed that probabilities alone may not be comprehensive enough to effectively capture expert knowledge. In many instances, it becomes essential to move beyond probabilities and consider the expert's level of confidence or assurance regarding the actual occurrence of an event. In line with this notion, we introduce the concept of fuzzy measure.

Definition 2.2. (Fuzzy measure) [33].
Let $\mathcal{X}$ be a σ-algebra on a non-empty finite set X. A function $\mu : \mathcal{X} \to [0, 1]$ is considered a fuzzy measure on the measurable space $(X, \mathcal{X})$ if and only if μ satisfies the following requirements:

(1) $\mu(\emptyset) = 0$, $\mu(X) = 1$,

(2) for any $A, B \in \mathcal{X}$ with $B \subseteq A$, it holds that $\mu(B) \leqslant \mu(A)$ (monotonicity).

The following is a generalization of both probability and fuzzy measures.

Definition 2.3. (Sugeno λ-measure) [33].
Let $X_i, i \in \{1, \cdots, n\}$, be pairwise disjoint sets and let $X = \bigcup_{i=1}^{n} X_i$. Suppose further that $\lambda \in (-1, +\infty)$. A Sugeno λ-measure is a function $g_\lambda : \mathcal{X} \to [0, 1]$ such that

1. $g_\lambda(X) = g_\lambda(\bigcup\limits_{i=1}^{n} X_i) = 1$,

2. For any $A, B \in \mathcal{X}$ with $A \cap B = \emptyset$ the following equality holds

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) \cdot g_\lambda(B) \tag{3}$$

Remark 1. [39] Let us observe that the parameter λ satisfies the following properties:

a)
$$\lambda + 1 = \prod_{i=1}^{n} [1 + \lambda g_\lambda(X_i)] \,. \tag{4}$$

b) If $\sum\limits_{i=1}^{n} g_\lambda(X_i) = 1$, then $\lambda = 0$ indicating that g_λ is an additive measure.

Example 2.1. Calculate the Sugeno λ-measure for the group of students in the functional equation class. Given the set of sets $X = X_1$(the set of students who scored 5 in functional equations) $\cup$ X_2(the set of students who scored 4 in functional equations) $\cup$ X_3(the set of students who scored 3 in functional equations) with fuzzy values as follows

$$g_\lambda(X_1) = 0.5,$$

$$g_\lambda(X_2) = 0.4,$$

$$g_\lambda(X_3) = 0.3.$$

First we calculate the value of λ using the equation; $\lambda+1 = \prod_{i=1}^{3}(1+\lambda g_\lambda(X_i))$. Simplifying the equation, we have, $\lambda + 1 = (0.5\lambda + 1)(0.4\lambda + 1)(0.3\lambda + 1)$, and so, $0.06\lambda^3 + 0.47\lambda^2 + 0.2\lambda = 0$, thus we obtain $\lambda \in \{0, -7.3818, -0.4516\}$. Since $\lambda \in (-1, \infty)$, we conclude that $\lambda = 0$ or $\lambda =$ -0.4516. Observing that $\sum_{i=1}^{3} g_\lambda(X_i) \neq 1$, we conclude that $\lambda \neq 0$. Therefore, we choose $\lambda =$ -0.4516. This choice of λ leads to the following measures;
$g_\lambda(X_1 \cup X_2) = g_\lambda(X_1) + g_\lambda(X_2) + \lambda \cdot g_\lambda(X_1) \cdot g_\lambda(X_2) = 0.8097$,
$g_\lambda(X_1 \cup X_3) = g_\lambda(X_1) + g_\lambda(X_3) + \lambda \cdot g_\lambda(X_1) \cdot g_\lambda(X_3) = 0.7323$,
$g_\lambda(X_2 \cup X_3) = g_\lambda(X_2) + g_\lambda(X_3) + \lambda \cdot g_\lambda(X_2) \cdot g_\lambda(X_3) = 0.6458$, and finally we have
$g_\lambda(X_1 \cup X_2 \cup X_3) = g_\lambda(X) = 1$.

Definition 2.4. (Sugeno Integral)[58].

Let μ be a fuzzy normalized measure defined on the set of sets $X = \bigcup_{i=1}^{n} X_i$, where X_i are pairwise disjoint sets. The Sugeno integral of a function $f : X \to [0, 1]$ with respect to the fuzzy measure μ is given by:

$$(S)\int f d\mu = [f(X_1) \wedge \mu(A_1)] \vee [f(X_2) \wedge \mu(A_2)] \vee \cdots \vee [f(X_n) \wedge \mu(A_n)] \quad (5)$$

Here, $\mu(A_1) = \mu(X_1 \cup X_2 \cup \ldots \cup X_n)$, $\mu(A_2) = \mu(X_2 \cup X_3 \cup \ldots \cup X_n)$, $\ldots$, $\mu(A_n) = \mu(X_n)$. The ranges $\{f(X_1), f(X_2), \ldots, f(X_n)\}$ are defined in ascending order as $f(X_1) \leqslant f(X_2) \leqslant \ldots \leqslant f(X_n)$.

Example 2.2. Consider the sets $X_i, i \in \{1, 2, 3\}$ with ranges given as $f(X_1) = 0.1, f(X_2) = 0.4, f(X_3) = 0.7$ such that the fuzzy measure is defined as

function	values	Normalized fuzzy measure
$\mu(\varnothing)$	0	0
$\mu(X_1)$	0.1	0.08
$\mu(X_2)$	0.3	0.23
$\mu(X_3)$	0.5	0.38
$\mu(X_1 \cup X_2)$	0.6	0.46
$\mu(X_2 \cup X_3)$	0.7	0.54
$\mu(X_1 \cup X_3)$	0.8	0.62
$\mu(X_1 \cup X_2 \cup X_3)$	1.3	1

To normalize the fuzzy measures, we divide all of them by the largest value in measure. The Sugeno integral is defined as

$$(S)\int f d\mu = [0.1 \wedge 1] \vee [0.4 \wedge 0.54] \vee [0.7 \wedge 0.38]$$

which finally gives us

$$(S)\int f d\mu = 0.4.$$

Example 2.3. Consider the sets $X_i, i \in \{1,2,3\}$ with ranges defined as $f(X_1) = 0.4, f(X_2) = 0.6, f(X_3) = 0.8$ and fuzzy values given as

$$g_\lambda(X_1)=0.6,$$
$$g_\lambda(X_2)=0.4,$$
$$g_\lambda(X_3)=0.3.$$

We begin by calculating λ using the equation; $\lambda + 1 = \prod_{i=1}^{3}(1 + \lambda g_\lambda(X_i))$. Simplifying the equation, we have, $\lambda+1 = (0.6\lambda+1)(0.4\lambda+1)(0.3\lambda+1)$, and so, $0.072\lambda^3 + 0.54\lambda^2 + 0.3\lambda = 0$, thus we get $\lambda \in \{0, -0.6042, -6.8958\}$. Since $\lambda \in (-1, \infty)$ we have that $\lambda = 0$ or λ = -0.6042. Observing that $\sum_{i=1}^{3} g_\lambda(X_i) \neq 1$, we conclude that $\lambda \neq 0$. Hence, we choose $\lambda = -0.6042$ and we calculate the following measures
$g_\lambda(X_1 \cup X_2) = g_\lambda(X_1) + g_\lambda(X_2) + \lambda g_\lambda(X_1) \cdot g_\lambda(X_2) = 0.855$,
$g_\lambda(X_1 \cup X_3) = g_\lambda(X_1) + g_\lambda(X_3) + \lambda g_\lambda(X_1) \cdot g_\lambda(X_3) = 0.7912$,
$g_\lambda(X_2 \cup X_3) = g_\lambda(X_2) + g_\lambda(X_3) + \lambda g_\lambda(X_2) \cdot g_\lambda(X_3) = 0.6275$, and finally
$g_\lambda(X_1 \cup X_2 \cup X_3) = g_\lambda(X) = 1$. Therefore, the Sugeno integral is

$$(S)\int f dg_\lambda = [0.4 \wedge 1] \vee [0.60 \wedge 0.6275] \vee [0.8 \wedge 0.3] = 0.6.$$

Definition 2.5. (Choquet Integral)[14].

Let μ be a fuzzy normalized measure defined on the set of sets $X = \bigcup_{i=1}^{n} X_i$, where X_i are pairwise disjoint sets. The Choquet integral of a function $f : X \to [0, \infty]$ with respect to the fuzzy measure μ is given by:

$$(C)\int_A f d\mu = \sum_{i=1}^{n}[f(X_i) - f(X_{i-1})]\mu(A_i), \tag{6}$$

Here, $\mu(A_1) = \mu(X_1 \cup X_2 \cup \ldots \cup X_n)$, $\mu(A_2) = \mu(X_2 \cup X_3 \cup \ldots \cup X_n)$, $\ldots$, $\mu(A_n) = \mu(X_n)$. The ranges $\{f(X_1), f(X_2), \ldots, f(X_n)\}$ are defined in ascending order as $f(X_1) \leqslant f(X_2) \leqslant \ldots \leqslant f(X_n)$ $f(X_0) = 0$.

Example 2.4. From example 2.6, we can equally compute Choquet integral as shown below,

$$\begin{aligned}(C)\int_A f dg_\lambda &= \sum_{i=1}^{3}[f(X_i) - f(X_{i-1})]g_\lambda(A_i)\\ &= [f(X_1) - f(X_0)]\cdot g_\lambda(A_1) + [f(X_2) - f(X_1)]\cdot g_\lambda(A_2) + [f(X_3) - f(X_2)]\cdot g_\lambda(A_3).\\ &= 1(0.4) + (0.6 - 0.4)0.6275 + (0.8 - 0.6)0.3 = 0.5855.\end{aligned}$$

We observe that Sugeno integral value for example 2.6 is higher than the Choquet integral. So, this suggests that the minimum value in the set of values is not too low. This is because the Sugeno integral is more sensitive to the minimum values in a set, while the Choquet integral is more sensitive to the relative importance of the different values in a set.

Taking into account Definition 2.3 and Remark 1, we introduce a modified version of the Sugeno λ-measure, which we refer to as the m_λ-measure which is helpful in computing fuzzy lambda measure for large domains.

Definition 2.6. (m_λ-measure)
Let $X_i, i \in \{1, \ldots, n\}$ be pairwise disjoint sets and let X represent the union of all these X_i sets. Suppose further that $\lambda \in (-1, +\infty)$, the function $m_\lambda : \mathcal{X} \to [0, 1]$, is referred to as an m_λ-measure if it satisfies the following conditions:

1) $m_\lambda(X) = m_\lambda(\bigcup_{i=1}^{n} X_i) = 1$

2)
$$m_\lambda(\bigcup_{i=1}^{m} X_i) = \sum_{j=1}^{m} \sum_{\hat{X}\in\sigma(j)} \lambda^{j-1} m_\lambda(\hat{X}) \tag{7}$$

here each $\hat{X} \in \sigma(j)$ has cardinality of j(card$(\hat{X})$ $=j$) and $\sigma(j)$ represents a collection of sets, where each member have exactly j number of disjoint set(s) and $m_\lambda(\hat{X}) = \prod_{\hat{X}_* \in \hat{X}} m_\lambda(\hat{X}_*)$.

2.2 Sugeno and Choquet integrals for nonnegative functions

It's well known that in case of classical measure (additive) measures, the Lebesgue integral is the usual definition of an integral with respect to measure, and it allows the computation of the expected value of random variables. Two concepts of integrals emerge; the one proposed by Choquet in 1953, and the one proposed by Sugeno in 1974 which define integral of a function with respect to a non-additive measure, i.e. capacity or game. Both are based on decumulative distribution function on the integrand with respect to the capacity.
Most of the other concepts of integral are also based on the decumulative function like the Shilkret integral, but other approaches are possible.

Definition 2.7. (Discrete Sugeno Integral)[58].
Let μ be a fuzzy normalized measure defined on the set of sets $X = \bigcup_{i=1}^{n} X_i$, where X_i are pairwise disjoint sets. The Sugeno integral of a function $f : X \to [0,1]$ with respect to the fuzzy measure μ is given by:

$$(S)\int f d\mu = [f(X_1) \wedge \mu(A_1)] \vee [f(X_2) \wedge \mu(A_2)] \vee \cdots \vee [f(X_n) \wedge \mu(A_n)] \quad (8)$$

Here, $\mu(A_1) = \mu(X_1 \cup X_2 \cup \ldots \cup X_n)$, $\mu(A_2) = \mu(X_2 \cup X_3 \cup \ldots \cup X_n)$, $\ldots$, $\mu(A_n) = \mu(X_n)$. The ranges $\{f(X_1), f(X_2), \ldots, f(X_n)\}$ are defined in ascending order as $f(X_1) \leqslant f(X_2) \leqslant \ldots \leqslant f(X_n)$.

Example 2.5. Consider the sets $X_i, i \in \{1,2,3\}$ with ranges given as $f(X_1) = 0.1, f(X_2) = 0.4, f(X_3) = 0.7$ such that the fuzzy measure is defined as

function	values	Normalized fuzzy measure
$\mu(\emptyset)$	0	0
$\mu(X_1)$	0.1	0.08
$\mu(X_2)$	0.3	0.23
$\mu(X_3)$	0.5	0.38
$\mu(X_1 \cup X_2)$	0.6	0.46
$\mu(X_2 \cup X_3)$	0.7	0.54
$\mu(X_1 \cup X_3)$	0.8	0.62
$\mu(X_1 \cup X_2 \cup X_3)$	1.3	1

To normalize the fuzzy measures, we divide all of them by the largest value in measure. The Sugeno integral is defined as

$$(S)\int f d\mu = [0.1 \wedge 1] \vee [0.4 \wedge 0.54] \vee [0.7 \wedge 0.38]$$

which finally gives us

$$(S)\int f d\mu = 0.4.$$

Example 2.6. Consider the sets $X_i, i \in \{1,2,3\}$ with ranges defined as $f(X_1) = 0.4, f(X_2) = 0.6, f(X_3) = 0.8$ and fuzzy values given as

$$g_\lambda(X_1)=0.6,$$
$$g_\lambda(X_2)=0.4,$$
$$g_\lambda(X_3)=0.3.$$

We begin by calculating λ using the equation; $\lambda + 1 = \prod_{i=1}^{3}(1 + \lambda g_\lambda(X_i))$. Simplifying the equation, we have $\lambda+1 = (0.6\lambda+1)(0.4\lambda+1)(0.3\lambda+1)$, and so, $0.072\lambda^3 + 0.54\lambda^2 + 0.3\lambda = 0$, thus we get $\lambda \in \{0, -0.6042, -6.8958\}$. Since $\lambda \in (-1, \infty)$ we have that $\lambda = 0$ or λ = -0.6042. Observing that $\sum_{i=1}^{3} g_\lambda(X_i) \neq 1$, we conclude that $\lambda \neq 0$. Hence, we choose $\lambda = -0.6042$ and we calculate the following measures
$g_\lambda(X_1 \cup X_2) = g_\lambda(X_1) + g_\lambda(X_2) + \lambda g_\lambda(X_1) \cdot g_\lambda(X_2) = 0.855$,
$g_\lambda(X_1 \cup X_3) = g_\lambda(X_1) + g_\lambda(X_3) + \lambda g_\lambda(X_1) \cdot g_\lambda(X_3) = 0.7912$,
$g_\lambda(X_2 \cup X_3) = g_\lambda(X_2) + g_\lambda(X_3) + \lambda g_\lambda(X_2) \cdot g_\lambda(X_3) = 0.6275$, and finally
$g_\lambda(X_1 \cup X_2 \cup X_3) = g_\lambda(X) = 1$. Therefore, the Sugeno integral is

$$(S)\int f dg_\lambda = [0.4 \wedge 1] \vee [0.60 \wedge 0.6275] \vee [0.8 \wedge 0.3] = 0.6.$$

Definition 2.8. (Choquet Integral)[14].
Let μ be a fuzzy normalized measure defined on the set of sets $X = \bigcup_{i=1}^{n} X_i$, where X_i are pairwise disjoint sets. The Choquet integral of a function $f : X \to [0, \infty]$ with respect to the fuzzy measure μ is given by:

$$(C)\int_A f d\mu = \sum_{i=1}^{n}[f(X_i) - f(X_{i-1})]\mu(A_i), \tag{9}$$

Here, $\mu(A_1) = \mu(X_1 \cup X_2 \cup \ldots \cup X_n)$, $\mu(A_2) = \mu(X_2 \cup X_3 \cup \ldots \cup X_n)$, $\ldots$, $\mu(A_n) = \mu(X_n)$. The ranges $\{f(X_1), f(X_2), \ldots, f(X_n)\}$ are defined in ascending order as $f(X_1) \leqslant f(X_2) \leqslant \ldots \leqslant f(X_n)$ $f(X_0) = 0$.

Example 2.7. From example 2.6, we can equally compute Choquet integral as shown below,

$$
\begin{aligned}
(C)\int_A f dg_\lambda &= \sum_{i=1}^{3}[f(X_i) - f(X_{i-1})]g_\lambda(A_i) \\
&= [f(X_1) - f(X_0)]\cdot g_\lambda(A_1) + [f(X_2) - f(X_1)]\cdot g_\lambda(A_2) + [f(X_3) - f(X_2)]\cdot g_\lambda(A_3) \\
&= 1(0.4) + (0.6 - 0.4)0.6275 + (0.8 - 0.6)0.3 = 0.5855.
\end{aligned}
$$

We observe that Sugeno integral value for example 2.6 is higher than the Choquet integral. So, this suggests that the minimum value in the set of values is not too low. This is because the Sugeno integral is more sensitive to the minimum values in a set, while the Choquet integral is more sensitive to the relative importance of the different values in a set.

Next we define the continuous Sugeno Integral. Suppose (X, Σ, μ) is a fuzzy measure space and that $\mathbf{F}$ is the class of all finite non-negative measurable functions defined on (X, Σ, μ).
Then for any $f \in \mathbf{F}$, we write $F_\alpha = \{x : f(x) \geqslant \alpha\}$ for $\alpha \in [0, \infty]$.

Definition 2.9 (Generalized Sugeno Integral)**.** The generalized Sugeno integral of $f \in \mathbf{F}$ on $A \in \Sigma$ is defined as

$$\int_{\circ, A} f d\mu = \sup_{t \geqslant 0}(t \circ \mu(A \cap \{f \geqslant t\})). \tag{10}$$

where $\{f \geqslant t\} = \{x \in X : f(x) \geqslant t\}$, μ is a monotone measure on Σ and $\circ$ is a non-decreasing binary map.
Commonly encountered examples of the generalized Sugeno integral include:

1. The Sugeno integral
$$\int_A f d\mu = \sup_{t \geqslant 0}(t \wedge \mu(A \cap \{f \geqslant t\})), \tag{11}$$
2. The Shilkret integral
$$\int_A f d\mu = \sup_{t \geqslant 0}(t.\mu(A \cap \{f \geqslant t\})),$$
3. The q-integral and the semi-normed fuzzy integral.

Here and subsequently, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Definition 2.10 ([64], Sugeno integral)**.** Let μ be a fuzzy measure. If $f \in \mathbf{F}$ and $A \in \Sigma$, then the Sugeno integral of f on A, with respect to the fuzzy measure μ is defined by

$$\int_A f d\mu = \bigvee_{\alpha \geqslant 0} [\alpha \wedge \mu(A \cap F_\alpha)]. \tag{12}$$

where $\bigvee$ denotes the operation sup, or upper bound. If $A = X$ then

$$\int_X f d\mu = \bigvee_{\alpha \geqslant 0} [\alpha \wedge \mu(f \geqslant \alpha)].$$

Example 2.8. Let $X = [0, 1], \mu = m^2$, where m is the Lebesgue measure, $f(x) = \frac{x}{2}$. We have

$$F_\alpha = \{x : f(x) \geqslant \alpha\} = [2\alpha, 1].$$

We only need to consider $\alpha \in [0, \frac{1}{2})$. So we have that

$$\int_A f d\mu = \sup_{\alpha \in [0, \frac{1}{2})} [\alpha \wedge (1 - 2\alpha)^2].$$

In the expression, $(1 - 2\alpha)^2$ is a decreasing continuous function of α when $\alpha \in [0, \frac{1}{2})$.

Hence, the supremum will be attained at the point which is the solution of $\alpha = (1 - 2\alpha)^2$, that is, at $\alpha = \frac{1}{4}$.

Consequently, we have

$$\int_A f d\mu = \frac{1}{4}.$$

Let us enumerate some properties of the Sugeno integral which will be useful in the sequel.

Proposition 1. *([64]) If μ is a fuzzy measure on X and $f, g \in \boldsymbol{F}, \alpha \in [0, \infty]$ then:*

1. $\int_A f d\mu \leqslant \mu(A) \iff \mu(A \cap F_\alpha) \leqslant \mu(A), \alpha \geqslant 0$
2. *If $\mathbb{1}_A$ is the characteristic function of A then* $\int_X f \mathbb{1}_A d\mu = \int_A f d\mu$.
3. $\int_A f d\mu > \alpha \iff \exists \beta > \alpha$ *such that* $\mu(A \cap F_\beta) > \alpha$.

4. *If* $\mu(A) < \infty$, *then* $\int_A f d\mu \geqslant \alpha \iff \mu(A \cap \{f \geqslant \alpha\}) \geqslant \alpha$.

5. *If* $\mu(A) < \infty$, *then* $\int_A f d\mu \leqslant \alpha \iff \mu(A \cap \{f \geqslant \alpha\}) \leqslant \alpha$.

6. *If* $f_1 \leqslant f_2$ *then* $\int_A f_1 d\mu \leqslant \int_A f_2 d\mu$.

7. $\int_A a d\mu = a \wedge \mu(A)$ *for any constant* $a \in [0, \infty]$.

Remark 2. Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap F_\alpha)$. Then, due to 4. and 5. of Proposition 1, we have that $F(\alpha) = \alpha \Rightarrow \int_A f d\mu = \alpha$. Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.
Here and subsequently, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

3 Hermite-Hadamard Inequality. ([44])

3.1 Introduction.

In this section, a computer program of solving inequalities of the form

$$a_1 f(\alpha_1 x + (1-\alpha_1)y) + \cdots + a_n f(\alpha_n x + (1-\alpha_n)y) \leqslant \frac{1}{y-x}\int_x^y f(t)dt \quad (13)$$

where a_i, α_i are non-negative real numbers and $\sum_{i=1}^{n} a_i = 1$ is presented, where the unknown function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous. This inequality includes, as particular cases, many well-known inequalities such as classical Hermite-Hadamard inequality in equation (1), Hermite-Hadamard inequalities of higher orders, Bullen inequality, and others. In the literature, there are known results where functional equations are solved with the use of computer programs. However such results were limited to solving equations(see for example [2],[3], [7], [13], [21], [28],[50],[51]). To the best of my knowledge, this is the first work where a computer program may be used to solve functional inequalities. The construction of our program is completely different from the methods used in the papers listed above and it is based on the results connected with the use of stochastic orderings tools. The idea to use stochastic orderings methods in the theory of inequalities was started by T. Rajba in [53] and then continued in many papers see for example [32],[38],[41][49],[52], [61]. The present approach is based on the ideas presented in the paper [62].
To present the solutions of (13) we need to use the higher-order convex functions. The simplest way to introduce the notion of higher-order convexity

is connected with divided differences of higher order which are recursively defined as follows:

$$f[x_1] = f(x_1)$$

and

$$f[x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_{n-1}] - f[x_2, \ldots, x_n]}{x_n - x_1}.$$

Now, we can present the definition of n−convex function. Let $I \subset \mathbb{R}$ be an interval. We say that function f is convex of order n if

$$f[x_1, \ldots, x_{n+2}] \geqslant 0,\ x_1, \ldots, x_{n+2} \in I.$$

Observe that the notion of convexity of higher orders naturally extends the usual convexity. First of all 0−convexity means non-decreasingness and 1−convexity is equivalent to the standard convexity. Moreover, in the class of n−times differentiable functions, the condition $f^{(n)} \geqslant 0$ is equivalent with $(n-1)$−convexity. For a survey of higher-order convex functions see for example [22].

3.2 A description of methods used in the computer program

The computer program solving (13) is divided into three steps. Each step is based on an important result. We explain these steps in this section.

3.2.1 Step 1

In this step, the main result from [6] is used. Let μ be a a non-zero bounded Borel (signed) measure on the interval $[0, 1]$, and let

$$\mu_n := \int_0^1 t^n d\mu(t),\ n = 0, 1, 2, \ldots\ .$$

Let n be the smallest non-negative integer such that $\mu_n \neq 0$. It was proved in [6] that $f : I \to \mathbb{R}$ is a continuous function satisfying the integral inequality

$$\int_0^1 f(x + t(y - x))d\mu(t) \geqslant 0 \tag{14}$$

then $\mu_n f$ is $(n-1)$−convex.

Take λ as a probabilistic measure that is uniformly distributed in the interval $[x, y]$ and define a measure ν by the formula

$$\nu = \sum_{i=1}^{n} a_i \delta_{\alpha_i x + (1-\alpha_i) y}$$

where $\delta_{\alpha_i x+(1-\alpha_i)y}$ is a measure concentrated in the point $\alpha_i x + (1 - \alpha_i)y$. Further, let $\mu = \lambda - \nu$, then our inequality (13) may be written as in the first step, we find an n such that the functions

$$1, x, \dots, x^n$$

satisfy (13) and $x \mapsto x^{n+1}$ does not satisfy it. Then we know that every continuous f satisfying (13) has to be $n-$convex (or $n-$concave).

3.2.2 Step 2

Here we use a simple sufficient condition for the higher order convex ordering from [16].

Theorem 3.1. *Let X and Y be two random variables such that*

$$\mathbb{E}[X^j - Y^j] = 0, \quad j = 1, 2, \dots, s.$$

If the distribution functions F_X, F_Y cross exactly $s-$times, at points $x_1, \dots, x_s$ and

$$(-1)^{s+1}(F_Y(t) - F_X(t)) \leqslant 0 \ \ \textit{for all } t \in [a, x_1]$$

then

$$\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$$

for all $s-$convex functions $f : \mathbb{R} \to \mathbb{R}$.

From now on it is clear that we need to pay attention to the cumulative distribution functions (shortly CDF) of the measures λ, ν. Observe that the CDF F_ν is a nondecreasing step function and F_λ is strictly increasing. Therefore there are two types of possible crossing points.

(i) We can have a crossing point in a node $\alpha_i x + (1 - \alpha_i)y$. It happens if

$$a_1 + \dots + a_{i-1} < F_\lambda(\alpha_i x + (1 - \alpha_i)y)$$

and

$$a_1 + \dots + a_i > F_\lambda(\alpha_i x + (1 - \alpha_i)y)$$

(ii) There may be a crossing point between the nodes. It is the case if

$$a_1 + \dots + a_{i-1} > F_\lambda(\alpha_{i-1} x + (1 - \alpha_{i-1})y)$$

and

$$a_1 + \dots + a_{i-1} < F_\lambda(\alpha_i x + (1 - \alpha_i)y).$$

The program checks the above conditions and counts the crossing points. If the obtained number of crossing points is equal to the number n obtained in the first step then the procedure is finished. A continuous function satisfies (13) if and only if it is n−convex (or n−concave, depending on the last sign of $F_\lambda - F_\nu$). If there are more crossing points than n, we need to proceed to the next step.

3.2.3 Step 3

If there are too many crossing points, then the convex ordering does not follow from the previous step. Thus a more precise result has to be used. Let X, Y be two random variables with the cumulative distribution functions F, G. Denote $F^{[0]}(t) := F(t)$ and $F^{[k]}(t) := \int_{-\infty}^{t} F^{[k-1]}(s)ds$. Using this notation, we formulate the following theorem from [57](see also [54]).

Theorem 3.2. *Let X, Y be two random variables with the cumulative distribution functions F, G, respectively. Then inequality*

$$\mathbb{E}[f(X)] \geqslant \mathbb{E}[f(Y)]$$

holds for all m−convex functions $f : \mathbb{R} \to \mathbb{R}$ if and only if the following two conditions are satisfied:

$$\mathbb{E}[X^k] = \mathbb{E}[Y^k], k = 1, \ldots, m \tag{15}$$

and

$$(-1)^m \big(G^{[m]}(t) - F^{[m]}(t)\big) \geqslant 0, for\ all\ t \in \mathbb{R}. \tag{16}$$

To use this theorem our program calculates the functions $F_\lambda^{[m]}$ and $F_\nu^{[m]}$ and checks the condition (16). It is first done in nodes: if the sign of the expression

$$F_\lambda^{[m]} - F_\nu^{[m]} \tag{17}$$

is different in any two nodes then we know that the condition (16) is not satisfied. However, if this sign is the same in all nodes it may still happen that it is different between the nodes. Therefore, we find the critical points of (17) and we check its values in these points. Then the procedure is finished. It should be emphasized that the form of (17) is different between different nodes and, therefore, the whole task requires simple but awkward calculations. Thus this is a very good task for the computer.

3.3 Computer code

The Python code provided below is designed to be executed exclusively within the Python Sagemath environment. It's important to highlight that the line **from sage.all import *** is functional solely in a Python environment with Sagemath installed. The subsequent code outlines our procedural steps as follows:

(a) Commence by importing the following Python libraries. These include sage.all, sys, sympy (Function, Symbol,symbols,cancel,N,sqrt,Rational), numpy, scipy.special (comb) and time. Subsequently, define the required variables and function. Here's the code snippet illustrating the process:

```
import sys
from sage.all import *
from sympy import Function Symbol,symbols,cancel,
N,sqrt,Rational
import numpy as np
from scipy.special import comb
import time
import sympy as sp
import matplotlib.pyplot as plt
x=Symbol('x')
t=Symbol('t')
y=Symbol('y')
f= Function('f')
```

(b) We have introduced a Python function referred to as **general_inequality**$(\cdot)$. This function encapsulates the entire Steps 1-3 and operates by taking a functional inequality in the form (13) as input. It's worth highlighting that the "$\leqslant \frac{1}{y-x}\int_x^y f(t)dt$" part of (13) remains consistent across all members of (13). Therefore, this component is embedded within the computer code and isn't required as input when executing the code. Consequently, the code is executed using the following command:

general_inequality $(a_1 f(\alpha_1 x + (1-\alpha_1)y) + \cdots + a_n f(\alpha_n x + (1-\alpha_n)y))$

The input is formulated in terms of the function f, and it's essential to recognize that any other representation will result in the generation of an error message.

It's important to highlight that the developed Python code is based on Python version 3.8 and Sagemath 9.1, along with all their necessary prerequisites. Both Python and Sagemath are open-source programming tools, and it's worth noting that some adjustments might be needed in the future to ensure compatibility with forthcoming versions of these software tools.

In the next subsection, we provide some results accompanied by graphs as well as some results without graphical representation. For a comprehensive view of all results including graphs, please refer to the GitHub URL above or feel free to reach out to the authors via email.

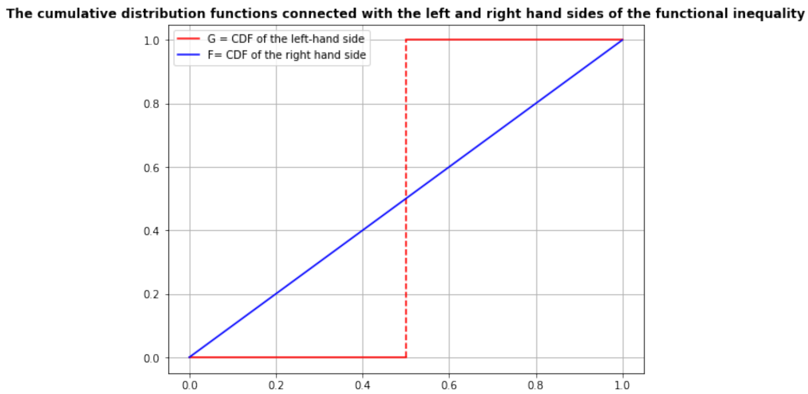
3.4 Results of the computer code

3.5 Functional inequalities considered by M. Bessenyei and Z. Páles

Example 3.1. (cf. Corollary 1 in [5]). Suppose that $f : [x, y] \to \mathbb{R}$,

$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:
general_inequality($f((x + y)/2)$)
OUTPUT:
By Step 1, every solution of the functional inequality must be a 1-concave function.

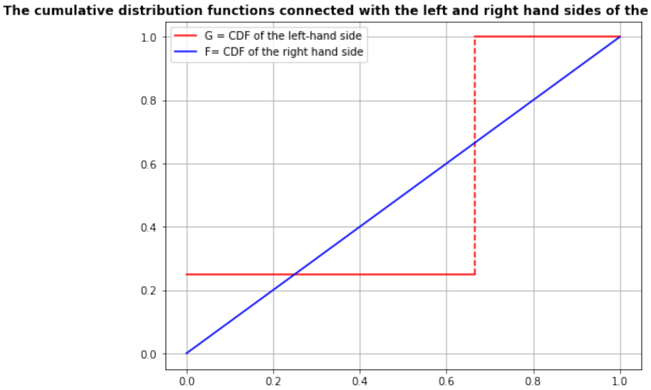


From Step 2, we obtain the number of the crossing points of these function to be 1. Since the number of crossing point is equal to 1 then we know that the functional inequality is satisfied by every 1-concave function f.

Example 3.2. (cf. Corollary 2 in [5]). Suppose that $f : [x, y] \to \mathbb{R}$,

$\left[\frac{1}{4}f(x) + \frac{3}{4}f\left(\frac{1}{3}x + \frac{2}{3}y\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:
general_inequality$((1/4) * f(x) + (3/4) * f((1/3) * x + (2/3) * y))$
OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 2. Since the number of crossing points is equal to 2 then we know that the functional inequality is satisfied by every 2-convex functions f.

Example 3.3. (cf. Corollary 3 in [5]). Suppose that $f : [x, y] \to \mathbb{R}$,

$\frac{1}{2}f\left(\frac{3+\sqrt{3}}{6}x + \frac{3-\sqrt{3}}{6}y\right) + \frac{1}{2}f\left(\frac{3-\sqrt{3}}{6}x + \frac{3+\sqrt{3}}{6}y\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:

$$\begin{aligned}\text{general\_inequality}((1/2) * (f(((3 + sqrt(3))/6) * x + ((3 - sqrt(3))/6) * y) \\ + f(((3 - sqrt(3))/6) * x + ((3 + sqrt(3))/6) * y)))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 3-convex functions.

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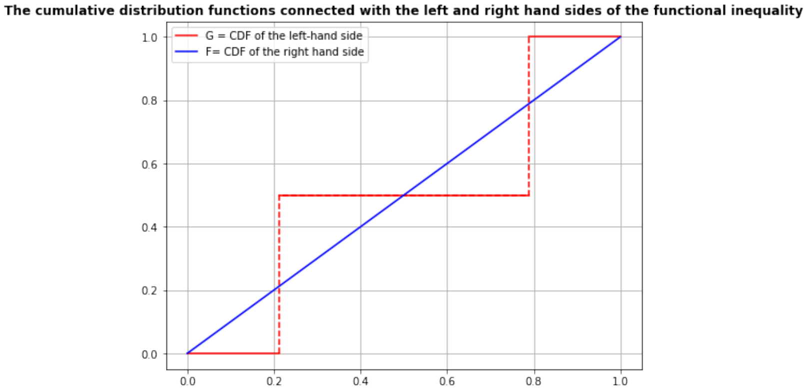


From Step 2, we obtain the number of the crossing points of these functions to be 3. Since the number of crossing points is equal to 3 then we know that the functional inequality is satisfied by every 3-convex functions f.

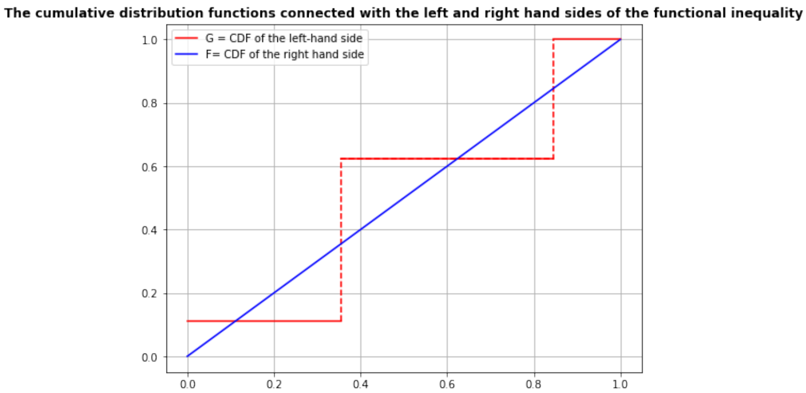
Example 3.4. (cf. Corollary 4 in [5]). Suppose that $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{9}f(x) + \frac{16+\sqrt{6}}{36}f\left(\frac{4+\sqrt{6}}{10}x + \frac{6-\sqrt{6}}{10}y\right) + \frac{16-\sqrt{6}}{36}f\left(\frac{4-\sqrt{6}}{10}x + \frac{6+\sqrt{6}}{10}y\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\begin{aligned}\text{general\_inequality}((1/9) * f(x) + ((16 + sqrt(6))/36) * (f(((4 + sqrt(6))/10) * x \\ + ((6 - sqrt(6))/10) * y)) + ((16 - sqrt(6))/36) \\ * (f(((4 - sqrt(6))/10) * x + ((6 + sqrt(6))/10) * y)))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 4-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4. Since the number of crossing points is equal to 4 then we know that

the functional inequality is satisfied by every 4-convex functions f.
- The time taken to solve the problem is 0.5 seconds -

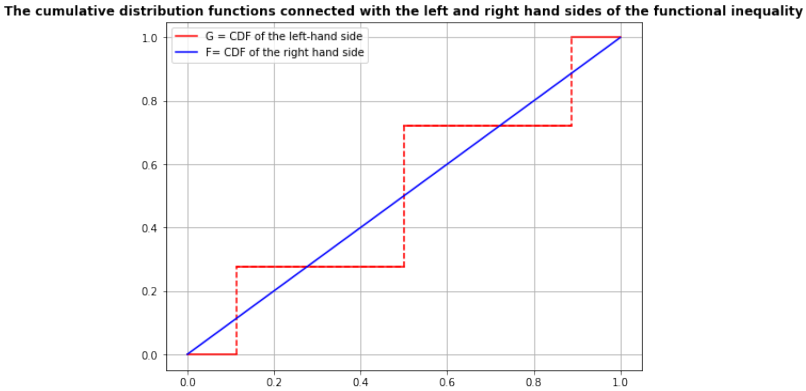
Example 3.5. (cf. Corollary 5 in [5]). Suppose that $f : [x, y] \to \mathbb{R}$,

$$\frac{5}{18} f\left(\frac{5+\sqrt{15}}{10}x + \frac{5-\sqrt{15}}{10}y\right) + \frac{4}{9} f\left(\frac{x+y}{2}\right) + \frac{5}{18} f\left(\frac{5-\sqrt{15}}{10}x + \frac{5+\sqrt{15}}{10}y\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT

$$\begin{aligned}\text{general\_inequality}((5/18) * (f(((5 + sqrt(15))/10) * x + ((5 - sqrt(15))/10) * y))\\ + (4/9) * (f((x + y)/2)) + (5/18) * (f(((5 - sqrt(15))/10) * x\\ + ((5 + sqrt(15))/10) * y)))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 5-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 5. Since the number of crossing points is equal to 5 then we know that the functional inequality is satisfied by every 5-convex functions f.
- The time taken to solve the problem is 0.53 seconds -

3.6 Functional inequalities considered by T. Szostok

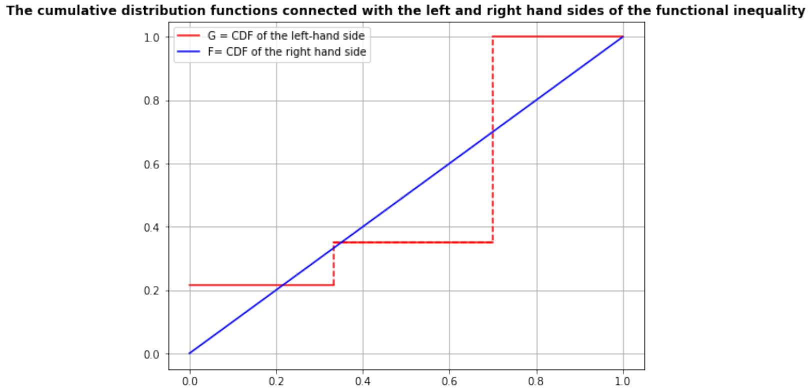
Example 3.6. (cf. Example 1 in [62]). Suppose that $f : [x, y] \to \mathbb{R}$,

$$\frac{3}{14} f(x) + \frac{3}{22} f\left(\frac{2}{3}x + \frac{1}{3}y\right) + \frac{50}{77} f\left(\frac{3}{10}x + \frac{7}{10}y\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\begin{aligned}\text{general\_inequality}((3/14) * f(x) + (3/22) * f((2/3) * x + (1/3) * y)\\ + (50/77) * f((3/10) * x + (7/10) * y))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied by every 2-convex functions f.
- The time taken to solve the problem is 1.03 seconds -

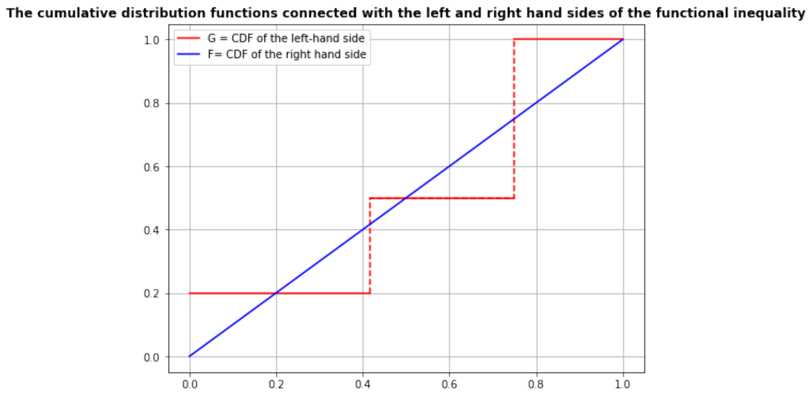
Example 3.7. (cf. Example 2 in [62]). Suppose that $f : [x, y] \to \mathbb{R}$,

$\frac{1}{5}f(x) + \frac{3}{10}f(\frac{7}{12}x + \frac{5}{12}y) + \frac{1}{2}f(\frac{1}{4}x + \frac{3}{4}y) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:

$$\text{general\_inequality}((1/5) * f(x) + (3/10) * f((7/12) * x + (5/12) * y) \\ + (1/2) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied by every 2-convex functions f.
- The time taken to solve the problem is 0.51 seconds -

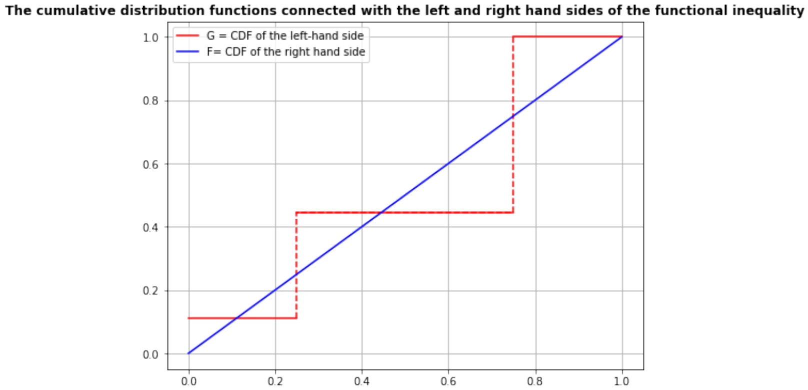
Example 3.8. (cf. Example 3 in [62]). Suppose that $f : [x, y] \to \mathbb{R}$,

$\frac{1}{9}f(x) + \frac{1}{3}f(\frac{3}{4}x + \frac{1}{4}y) + \frac{5}{9}f(\frac{1}{4}x + \frac{3}{4}y) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:

$$\text{general\_inequality}((1/9) * f(x) + (1/3) * f((3/4) * x + (1/4) * y) + (5/9) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



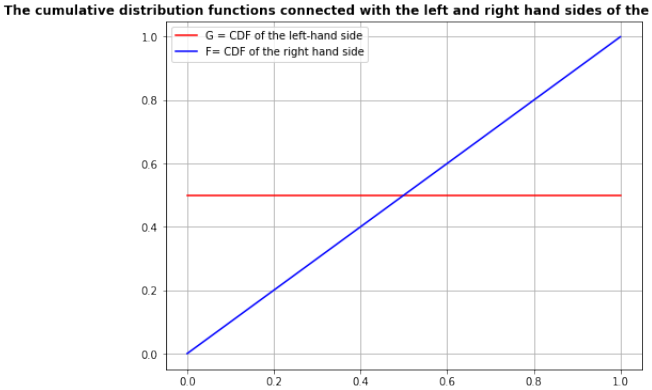
From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied by every 2-convex functions f.
- The time taken to solve the problem is 0.39 seconds -

3.7 Functional inequalities stemming from the known closed quadrature rules

Example 3.9. The 2-point rule is called the trapezoidal rule. Let $f : [x, y] \to \mathbb{R}$,

$\left[\frac{1}{2}f(x) + \frac{1}{2}f(y)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:
general_inequality$((1/2)*f(x)+(1/2)*f(y))$
OUTPUT:
By Step 1, every solution of the functional inequality must be a 1-concave function.

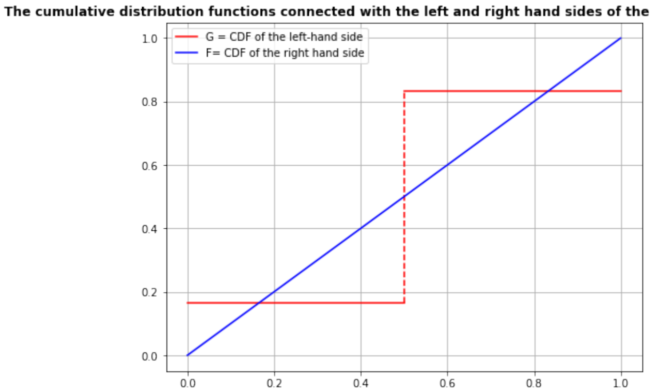


From Step 2, we obtain the number of the crossing points of these functions to be 1. Since the number of crossing points is equal to 1 then we know that the functional inequality is satisfied by every 1-concave function f.
- The time taken to solve the problem is 0.2 seconds -

Example 3.10. The 3-point rule is known as Simpson's rule. Let $f:[x,y]\to\mathbb{R}$,

$$\left[\frac{1}{6}f(x)+\frac{2}{3}f\left(\frac{x+y}{2}\right)+\frac{1}{6}f(y)\right]\leqslant\frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:
general_inequality$((1/6)*f(x)+(2/3)*f((x+y)*(1/2))+(1/6)*f(y))$
OUTPUT:
By Step 1, every solution of the functional inequality must be a 3-concave functions.



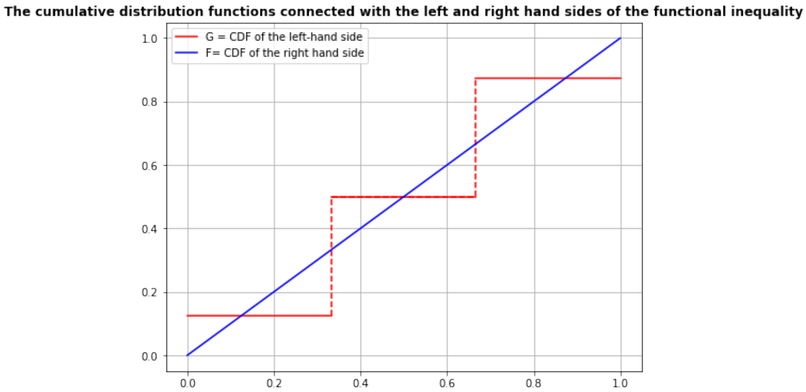
From Step 2, we obtain the number of the crossing points of these functions

to be 3. Since the number of crossing points is equal to 3 then we know that the functional inequality is satisfied by every 3-concave functions f.
- The time taken to solve the problem is 0.25 seconds -

Example 3.11. The 4-point closed rule is Simpson's 3/8 rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{8}\left[f(x) + 3f\left(\frac{x+2y}{3}\right) + 3f\left(\frac{2x+y}{3}\right) + f(y)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:
general_inequality$((1/8) * (f(x) + 3 * f((x + 2 * y) * (1/3)) + 3 * f((2 * x + y) * (1/3)) + f(y)))$
OUTPUT:
By Step 1, every solution of the functional inequality must be a 3-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 5.
By Step 3, the functional inequality is satisfied by every 3-concave function f.
- The time taken to solve the problem is 0.83 seconds -

Example 3.12. The 5-point closed rule is Boole's rule. Let $f : [x, y] \to \mathbb{R}$,

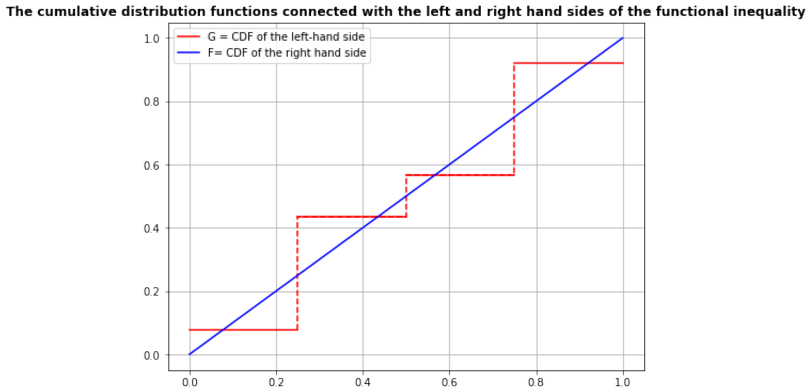
$$\frac{1}{90}\left[7f(x) + 32f\left(\frac{3x+y}{4}\right) + 12f\left(\frac{x+y}{2}\right) + 32f\left(\frac{x+3y}{4}\right) + 7f(y)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

general_inequality$((1/90) * (7 * f(x) + 32 * f((3/4) * x + (1/4) * y)$
$+ 12 * f((x + y)/2) + 32 * f((1/4) * x + (3/4) * y) + 7 * f(y)))$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 5-concave

functions.



From Step 2, we obtain the number of the crossing points of these functions to be 7.

By Step 3, the functional inequality is satisfied by every 5-concave functions f.

- The time taken to solve the problem is 6.69 seconds -

Example 3.13. The 6-point closed rule. Let $f : [x, y] \to \mathbb{R}$,

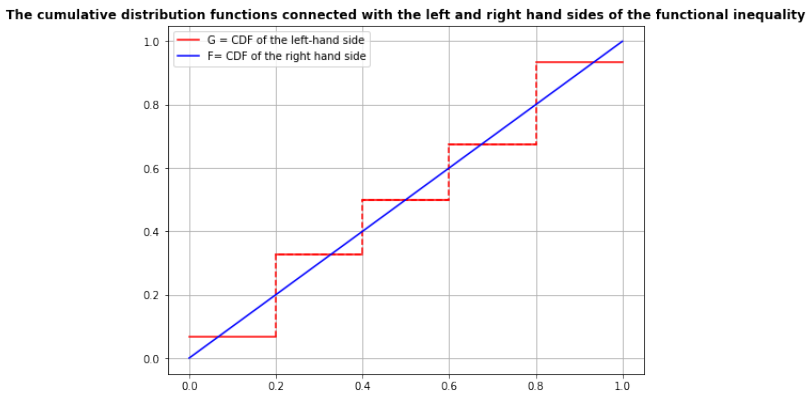
$[\frac{1}{288}\left[19f(x) + 75f\left(\frac{4x+y}{5}\right) + 50f\left(\frac{3x+2y}{5}\right) + 50f\left(\frac{2x+3y}{5}\right) + 75f\left(\frac{x+4y}{5}\right) + 19f(y)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:

$$\begin{aligned}\text{general\_inequality}((1/288) * (19 * f(x) &+ 75 * f((4/5) * x + (1/5) * y)\\ &+ 50 * f((3/5) * x + (2/5) * y) + 50 * f((2/5) * x + (3/5) * y)\\ &+ 75 * f((1/5) * x + (4/5) * y) + 19 * f(y)))\end{aligned}$$

OUTPUT:

By Step 1, every solution of the functional inequality must be a 5-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 9.
By Step 3, the functional inequality is satisfied by every 5-concave functions f.
- The time taken to solve the problem is 18.22 seconds -

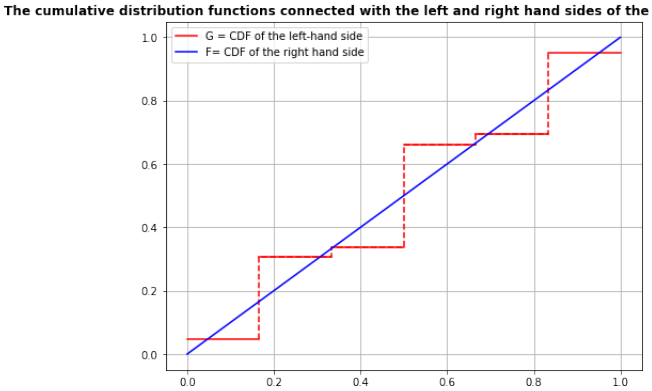
Example 3.14. The 7-point closed rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{840}[41f(x) + 216f\left(\frac{5x+y}{6}\right) + 27f\left(\frac{4x+2y}{6}\right) + 272f\left(\frac{3x+3y}{6}\right) + 27f\left(\frac{2x+4y}{6}\right) + 216f\left(\frac{x+5y}{6}\right) + 41f(y)] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\begin{aligned}\text{general\_inequality}((1/840) * (41 * f(x) + 216 * f((5/6) * x + (1/6) * y) \\ + 27 * f((4/6) * x + (2/6) * y) + 272 * f((3/6) * x + (3/6) * y) \\ + 27 * f((2/6) * x + (4/6) * y) + 216 * f((1/6) * x + (5/6) * y) \\ + 41 * f(y)))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 7-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 11.
By Step 3, the functional inequality is satisfied by every 7-concave functions f.
- The time taken to solve the problem is 61.2 seconds -

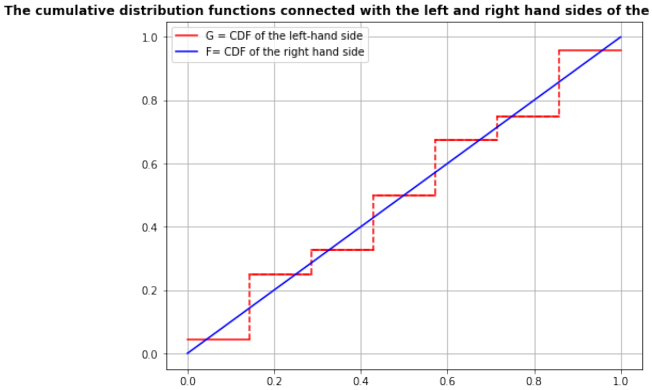
Example 3.15. The 8-point closed rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{17280}[751f(x) + 3577f\left(\frac{6x+y}{7}\right) + 1323f\left(\frac{5x+2y}{7}\right) + 2989f\left(\frac{4x+3y}{7}\right) + 2989f\left(\frac{3x+4y}{7}\right) + 1323f\left(\frac{2x+5y}{7}\right) + 3577f\left(\frac{x+6y}{7}\right) + 751f(y)] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/17280)*(751*f(x)+3577*f((6/7)*x+(1/7)*y) + 1323*f((5/7)*x+(2/7)*y)+2989*f((4/7)*x+(3/7)*y) + 2989*f((3/7)*x+(4/7)*y)+1323*f((2/7)*x+(5/7)*y) + 3577*f((1/7)*x+(6/7)*y)+751*f(y)))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 7-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 13.
By Step 3, the functional inequality is satisfied by every 7-concave functions f.
- The time taken to solve the problem is 74.09 seconds -

Example 3.16. The 9-point closed rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{28350}[989f(x)+5888f\left(\frac{7x+y}{8}\right)-928f\left(\frac{6x+2y}{8}\right)+10496f\left(\frac{5x+3y}{8}\right)-4540f\left(\frac{4x+4y}{8}\right) + 10496f\left(\frac{3x+5y}{8}\right)-928f\left(\frac{2x+6y}{8}\right)+5888f\left(\frac{x+7y}{8}\right)+989f(y)] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/28350)*(989*f(x)+5888*f((7/8)*x+(1/8)*y) - 928*f((6/8)*x+(2/8)*y)+10496*f((5/8)*x+(3/8)*y) - 4540*f((4/8)*x+(4/8)*y)+10496*f((3/8)*x+(5/8)*y) - 928*f((2/8)*x+(6/8)*y)+5888*f((1/8)*x+(7/8)*y) + 989*f(y)))$$

OUTPUT:
'This equation does not belong to class (1)'

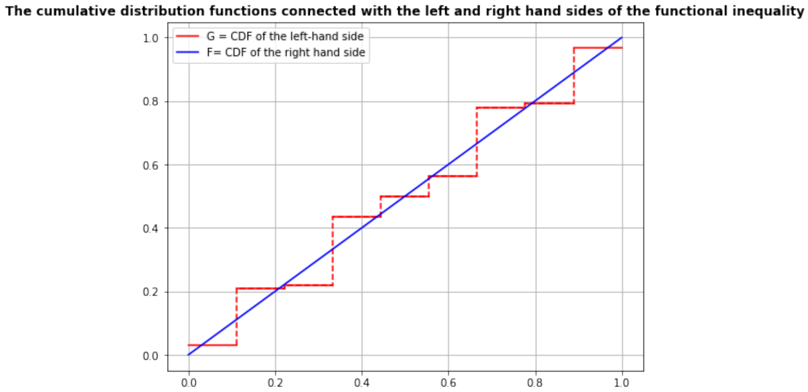
Example 3.17. The 10-point closed rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{89600}[2857(f(x) + f(y)) + 15741\left(f\left(\frac{8x+y}{9}\right) + f\left(\frac{x+8y}{9}\right)\right)$$
$$+ 1080\left(f\left(\frac{7x+2y}{9}\right) + f\left(\frac{2x+7y}{9}\right)\right) + 19344\left(f\left(\frac{6x+3y}{9}\right) + f\left(\frac{3x+6y}{9}\right)\right)$$
$$+ 5778\left(f\left(\frac{5x+4y}{9}\right) + f\left(\frac{4x+5y}{9}\right)\right)] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\begin{aligned}\text{general\_inequality}((1/89600) * (2857 * (f(x) + f(y)) \\ + 15741 * (f((8/9) * x + (1/9) * y) + f((1/9) * x + (8/9) * y)) \\ + 1080 * (f((7/9) * x + (2/9) * y) + f((2/9) * x + (7/9) * y)) \\ + 19344 * (f((6/9) * x + (3/9) * y) + f((3/9) * x + (6/9) * y)) \\ + 5778 * (f((5/9) * x + (4/9) * y) + f((4/9) * x + (5/9) * y))))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 9-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 13.
By Step 3, the functional inequality is satisfied by every 9-concave functions f.
- The time taken to solve the problem is 245.0 seconds -

Example 3.18. The 11-point closed rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{598752}[16067(f(x) + f(y)) + 106300\left(f\left(\frac{9x+y}{10}\right) + f\left(\frac{x+9y}{10}\right)\right)$$
$$- 48525\left(f\left(\frac{8x+2y}{10}\right) + f\left(\frac{2x+8y}{10}\right)\right) + 272400\left(f\left(\frac{7x+3y}{10}\right) + f\left(\frac{3x+7y}{10}\right)\right)$$
$$- 260550\left(f\left(\frac{6x+4y}{10}\right) + f\left(\frac{4x+6y}{10}\right)\right) + 427368 f\left(\frac{5x+5y}{10}\right)] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/598752)*(16067*(f(x)+f(y)) \\ +106300*(f((9/10)*x+(1/10)*y)+f((1/10)*x+(9/10)*y)) \\ -48525*(f((8/10)*x+(2/10)*y)+f((2/10)*x+(8/10)*y)) \\ +272400*(f((7/10)*x+(3/10)*y)+f((3/10)*x+(7/10)*y)) \\ -260550*(f((6/10)*x+(4/10)*y)+f((4/10)*x+(6/10)*y)) \\ +427368*f((5/10)*x+(5/10)*y)))$$

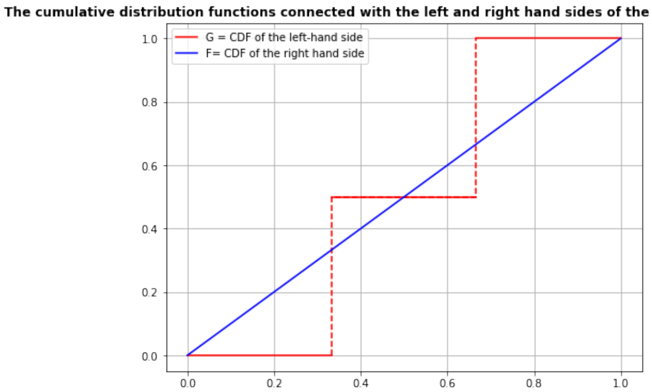
OUTPUT:
'This equation does not belong to class (1)'

3.8 Functional inequalities stemming from the known open Newton-Cotes rules

Example 3.19. The 2-point rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{2}f\left(\frac{2}{3}x+\frac{1}{3}y\right)+\frac{1}{2}f\left(\frac{1}{3}x+\frac{2}{3}y\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:
$\text{general\_inequality}((1/2)*f((2/3)*x+(1/3)*y)+(1/2)*f((1/3)*x+(2/3)*y))$
OUTPUT:
By Step 1, every solution of the functional inequality must be a 1-convex function.



From Step 2, we obtain the number of the crossing points of these functions to be 3.
By Step 3, the functional inequality is satisfied by every 1-convex function f.
- The time taken to solve the problem is 0.3 seconds -

Example 3.20. The 3-point rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{3}\left[2f\left(\frac{3}{4}x + \frac{1}{4}y\right) - f\left(\frac{2}{4}x + \frac{2}{4}y\right) + 2f\left(\frac{1}{4}x + \frac{3}{4}y\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/3) * (2 * f((3/4) * x + (1/4) * y) - f((2/4) * x + (2/4) * y) + 2 * f((1/4) * x + (3/4) * y)))$$

OUTPUT:
'This equation does not belong to class (1)'

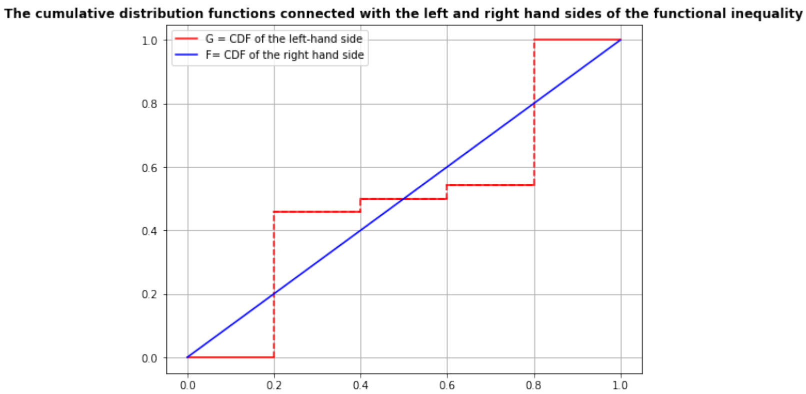
Example 3.21. The 4-point rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{24}\left[11f\left(\frac{4}{5}x + \frac{1}{5}y\right) + f\left(\frac{3}{5}x + \frac{2}{5}y\right) + f\left(\frac{2}{5}x + \frac{3}{5}y\right) + 11f\left(\frac{1}{5}x + \frac{4}{5}y\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/24) * (11 * f((4/5) * x + (1/5) * y) + f((3/5) * x + (2/5) * y) + f((2/5) * x + (3/5) * y) + 11 * f((1/5) * x + (4/5) * y)))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 3-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 3. Since the number of crossing point is equal to 3 then we know that the functional inequality is satisfied by every 3-convex functions f.
- The time taken to solve the problem is 0.37 seconds -

Example 3.22. The 5-point rule. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{20}\left[11f\left(\frac{5}{6}x + \frac{1}{6}y\right) - 14f\left(\frac{4}{6}x + \frac{2}{6}y\right) + 26f\left(\frac{3}{6}x + \frac{3}{6}y\right) - 14f\left(\frac{2}{6}x + \frac{4}{6}y\right) + 11f\left(\frac{1}{6}x + \frac{5}{6}y\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/20)*(11*f((5/6)*x+(1/6)*y)-14*f((4/6)*x+(2/6)*y)$$
$$+26*f((3/6)*x+(3/6)*y)-14*f((2/6)*x+(4/6)*y)$$
$$+11*f((1/6)*x+(5/6)*y)))$$

OUTPUT:
'This equation does not belong to class (1)'

Example 3.23. The 6-point rule. Let $f:[x,y]\to\mathbb{R}$,

$$\frac{1}{1440}[611f\left(\frac{6}{7}x+\frac{1}{7}y\right)-453f\left(\frac{5}{7}x+\frac{2}{7}y\right)+562f\left(\frac{4}{7}x+\frac{3}{7}y\right)+562f\left(\frac{3}{7}x+\frac{4}{7}y\right)$$
$$-453f\left(\frac{2}{7}x+\frac{5}{7}y\right)+611f\left(\frac{1}{7}x+\frac{6}{7}y\right)]\leqslant\frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/1440)*(611*f((6/7)*x+(1/7)*y)$$
$$-453*f((5/7)*x+(2/7)*y)+562*f((4/7)*x+(3/7)*y)$$
$$+562*f((3/7)*x+(4/7)*y)-453*f((2/7)*x+(5/7)*y)$$
$$+611*f((1/7)*x+(6/7)*y)))$$

OUTPUT:
'This equation does not belong to class (1)'

Example 3.24. The 7-point rule. Let $f:[x,y]\to\mathbb{R}$,

$$\frac{1}{945}[460f\left(\frac{7}{8}x+\frac{1}{8}y\right)-954f\left(\frac{6}{8}x+\frac{2}{8}y\right)+2196f\left(\frac{5}{8}x+\frac{3}{8}y\right)-2459f\left(\frac{4}{8}x+\frac{4}{8}y\right)$$
$$+2196f\left(\frac{3}{8}x+\frac{5}{8}y\right)-954f\left(\frac{2}{8}x+\frac{6}{8}y\right)+460f\left(\frac{1}{8}x+\frac{7}{8}y\right)]\leqslant\frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/945)*(460*f((7/8)*x+(1/8)*y)$$
$$-954*f((6/8)*x+(2/8)*y)+2196*f((5/8)*x+(3/8)*y)$$
$$-2459*f((4/8)*x+(4/8)*y)+2196*f((3/8)*x+(5/8)*y)$$
$$-954*f((2/8)*x+(6/8)*y)+460*f((1/8)*x+(7/8)*y)))$$

OUTPUT:
'This equation does not belong to class (1)'

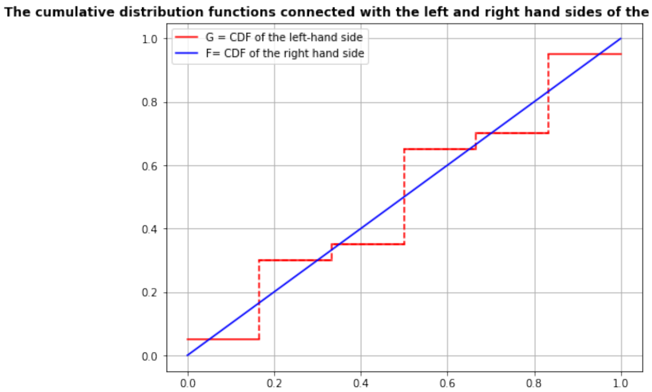
Example 3.25. Weddle's rule. Let $f:[x,y]\to\mathbb{R}$,

$$\frac{1}{20}[f(x)+5f\left(\frac{5}{6}x+\frac{1}{6}y\right)+f\left(\frac{4}{6}x+\frac{2}{6}y\right)+6f\left(\frac{3}{6}x+\frac{3}{6}y\right)+f\left(\frac{2}{6}x+\frac{4}{6}y\right)+5f\left(\frac{1}{6}x+\frac{5}{6}y\right)$$
$$+f(y)]\leqslant\frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/20)*(f(x)+5*f((5/6)*x+(1/6)*y)+f((4/6)*x+(2/6)*y) \\ +6*f((3/6)*x+(3/6)*y)+f((2/6)*x+(4/6)*y) \\ +5*f((1/6)*x+(5/6)*y)+f(y)))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 5-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 11.
By Step 3, the functional inequality is satisfied by every 5-concave functions f.
- The time taken to solve the problem is 16.19 seconds -

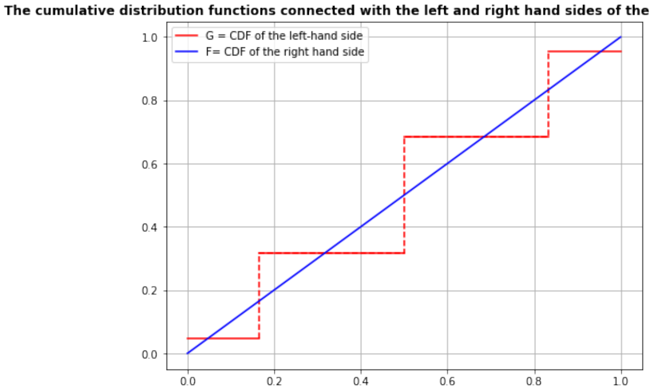
Example 3.26. Hardy's rule. Let $f : [x, y] \to \mathbb{R}$,

$\frac{1}{600}\left[28f(x) + 162f\left(\frac{5}{6}x + \frac{1}{6}y\right) + 220f\left(\frac{3}{6}x + \frac{3}{6}y\right) + 162f\left(\frac{1}{6}x + \frac{5}{6}y\right) + 28f(y)\right]$
$\leqslant \frac{1}{y-x}\int_x^y f(t)dt$

INPUT:

$$\text{general\_inequality}((1/600)*(28*f(x)+162*f((5/6)*x+(1/6)*y) \\ +220*f((3/6)*x+(3/6)*y) \\ +162*f((1/6)*x+(5/6)*y)+28*f(y)))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 5-concave functions.



From Step 2, we obtain the number of the crossing points of these functions to be 7.
By Step 3, the functional inequality is satisfied by every 5-concave functions f.
- The time taken to solve the problem is 11.33 seconds -

Example 3.27. A 3-point extended rule for odd n. Applying Simpson's 3/8 rule and Simpson's rule (3-point) twice, and adding gives. Let $f : [x, y] \to \mathbb{R}$,

$$\frac{1}{10}\Big[\frac{3}{8}f(x)+\frac{9}{8}f\left(\frac{9}{10}x+\frac{1}{10}y\right)+\frac{9}{8}f\left(\frac{8}{10}x+\frac{2}{10}y\right)+\frac{17}{24}f\left(\frac{7}{10}x+\frac{3}{10}y\right)+\frac{4}{3}f\left(\frac{6}{10}x+\frac{4}{10}y\right)$$
$$+\frac{2}{3}f\left(\frac{5}{10}x+\frac{5}{10}y\right)+\frac{4}{3}f\left(\frac{4}{10}x+\frac{6}{10}y\right)+\frac{17}{24}f\left(\frac{3}{10}x+\frac{7}{10}y\right)+\frac{9}{8}f\left(\frac{2}{10}x+\frac{8}{10}y\right)+\frac{9}{8}f\left(\frac{1}{10}x+\frac{9}{10}y\right)$$
$$+\frac{3}{8}f(y)\Big] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\begin{aligned}\text{general\_inequality}((1/10)*((3/8)*f(x)&+(9/8)*f((9/10)*x+(1/10)*y)\\&+(9/8)*f((8/10)*x+(2/10)*y)\\&+(17/24)*f((7/10)*x+(3/10)*y)\\&+(4/3)*f((6/10)*x+(4/10)*y)\\&+(2/3)*f((5/10)*x+(5/10)*y)\\&+(4/3)*f((4/10)*x+(6/10)*y)\\&+(17/24)*f((3/10)*x+(7/10)*y)\\&+(9/8)*f((2/10)*x+(8/10)*y)\\&+(9/8)*f((1/10)*x+(9/10)*y)+(3/8)*f(y)))\end{aligned}$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 3-concave functions.

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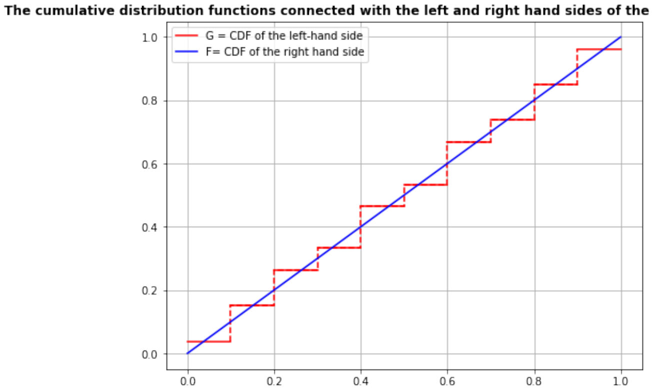


From Step 2, we obtain the number of the crossing points of these functions to be 19.
By Step 3, the functional inequality is satisfied by every 3-concave functions f.
- The time taken to solve the problem is 4.23 seconds -

3.9 Higher-order convex function of the functional inequality with negative results

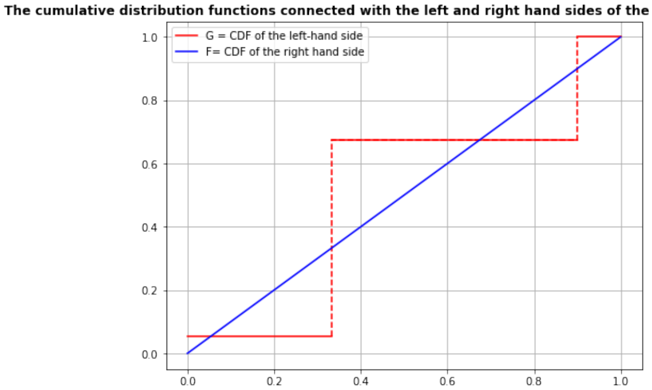
Example 3.28. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{1}{18}f(x) + \frac{21}{34}f\left(\frac{2x+y}{3}\right) + \frac{50}{153}f\left(\frac{x+9y}{10}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/18) * f(x) + (21/34) * f((2 * x + y)/3) + (50/153) * f((x + 9 * y)/10))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.54 seconds -

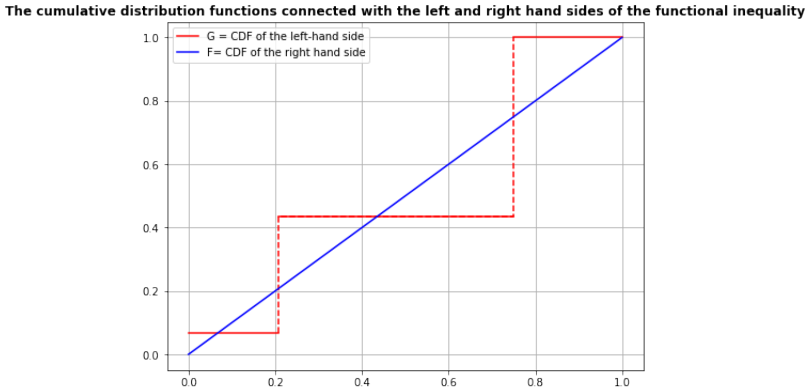
Example 3.29. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{1}{15}f(x) + \frac{24}{65}f\left(\frac{19x+5y}{24}\right) + \frac{22}{39}f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/15) * f(x) + (24/65) * f((19 * x + 5 * y)/24) + (22/39) * f((x + 3 * y)/4))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.37 seconds -

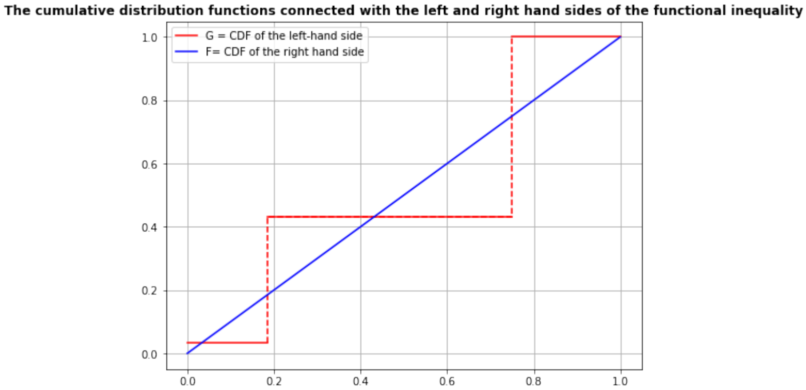
Example 3.30. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{1}{30}f(x) + \frac{243}{610}f\left(\frac{22x+5y}{27}\right) + \frac{104}{183}f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/30) * f(x) + (243/610) * f((22/27) * x + (5/27) * y) + (104/183) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.35 seconds -

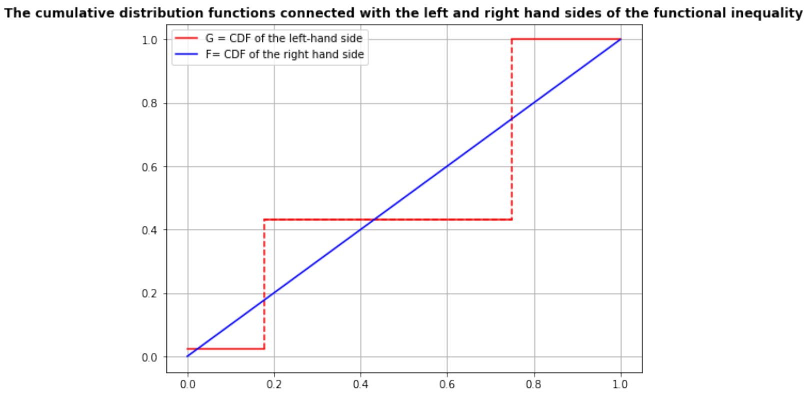
Example 3.31. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{1}{45} f(x) + \frac{49}{120} f\left(\frac{23x+5y}{28}\right) + \frac{41}{72} f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/45) * f(x) + (49/120) * f((23/28) * x + (5/28) * y) + (41/72) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.35 seconds -

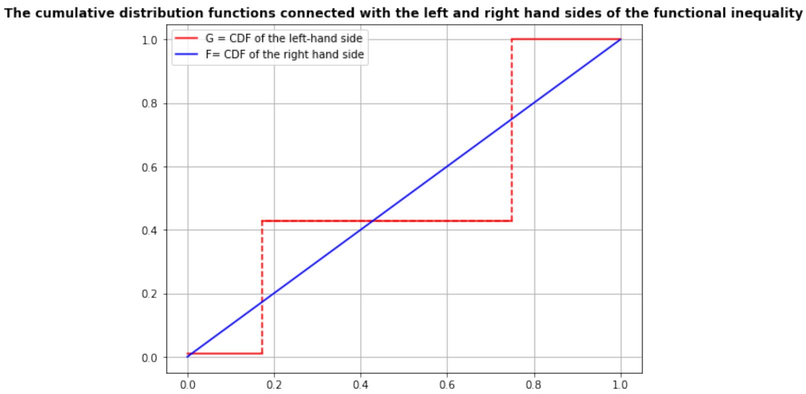
Example 3.32. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{1}{90}f(x) + \frac{841}{2010}f\left(\frac{24x+5y}{29}\right) + \frac{344}{603}f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((1/90) * f(x) + (841/2010) * f((24/29) * x + (5/29) * y) + (344/603) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.36 seconds -

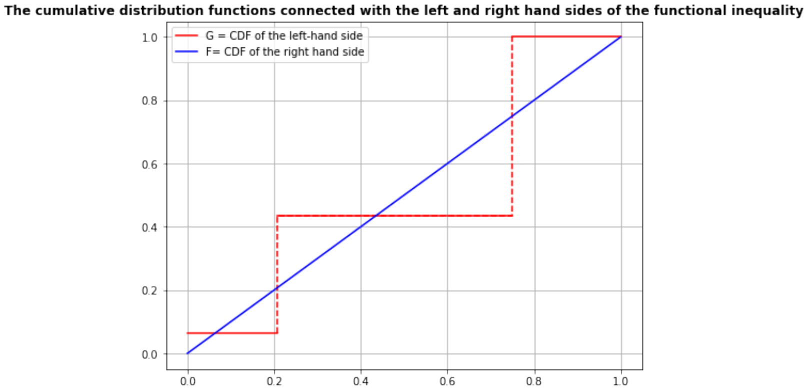
Example 3.33. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{7}{108}f(x) + \frac{841}{2268}f\left(\frac{23x+6y}{29}\right) + \frac{320}{567}f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((7/108) * f(x) + (841/2268) * f((23/29) * x + (6/29) * y) + (320/567) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.33 seconds -

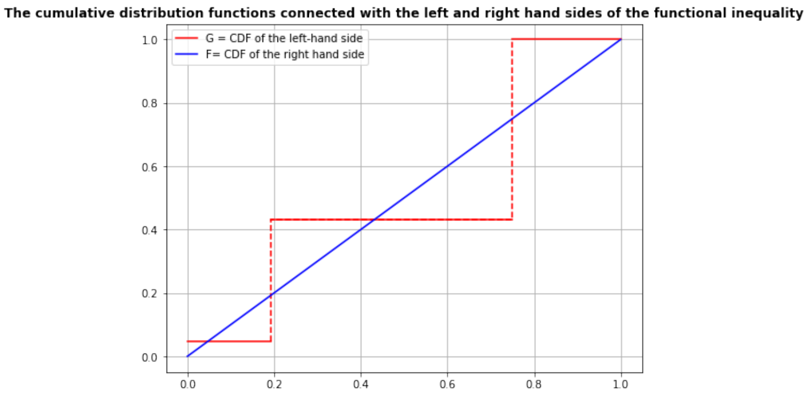
Example 3.34. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{5}{108} f(x) + \frac{961}{2484} f\left(\frac{25x+6y}{31}\right) + \frac{352}{621} f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x} \int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((5/108) * f(x) + (961/2484) * f((25/31) * x + (6/31) * y) + (352/621) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.41 seconds -

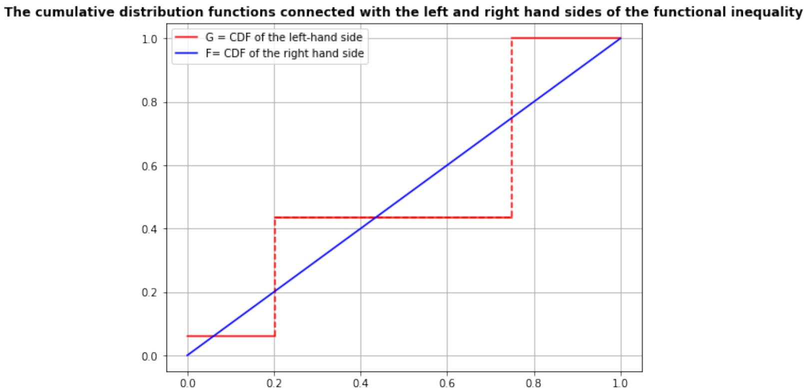
Example 3.35. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{13}{216} f(x) + \frac{3481}{9288} f\left(\frac{47x+12y}{59}\right) + \frac{656}{1161} f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((13/216) * f(x) + (3481/9288) * f((47/59) * x + (12/59) * y) + (656/1161) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.33 seconds -

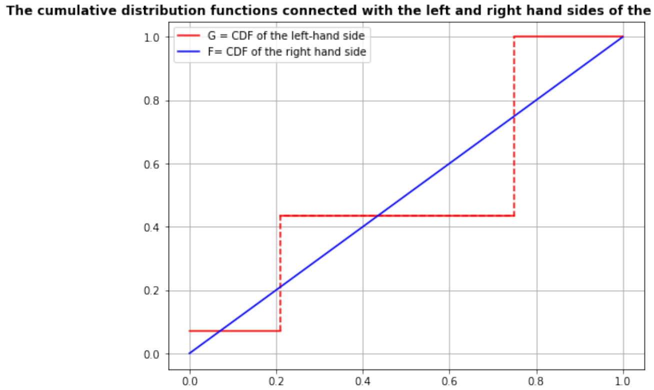
Example 3.36. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{29}{414} f(x) + \frac{11881}{32430} f\left(\frac{86x+23y}{109}\right) + \frac{1192}{2115} f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((29/414) * f(x) + (11881/32430) * f((86/109) * x + (23/109) * y) + (1192/2115) * f((1/4) * x + (3/4) * y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.4 seconds -

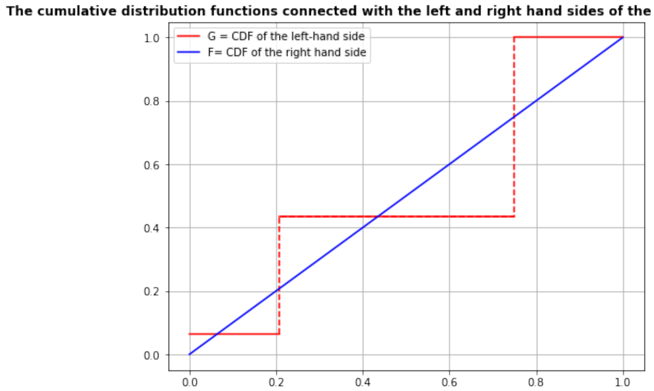
Example 3.37. Let $f : [x, y] \to \mathbb{R}$,

$$\left[\frac{3}{46}f(x) + \frac{4107}{11086}f\left(\frac{88x+23y}{111}\right) + \frac{136}{241}f\left(\frac{x+3y}{4}\right)\right] \leqslant \frac{1}{y-x}\int_x^y f(t)dt$$

INPUT:

$$\text{general\_inequality}((3/46)*f(x) + (4107/11086)*f((88/111)*x + (23/111)*y) + (136/241)*f((1/4)*x + (3/4)*y))$$

OUTPUT:
By Step 1, every solution of the functional inequality must be a 2-convex functions.



From Step 2, we obtain the number of the crossing points of these functions to be 4.
By Step 3, the functional inequality is satisfied neither by all 2-convex functions nor by all 2-concave functions.
- The time taken to solve the problem is 0.38 seconds -

Remark 3. It should be clearly stated that in the case where we get the negative result, we do not know how the solutions of (13) can be characterized. The only thing we know is that there exist $n-$convex functions which do not satisfy (13).

Remark 4. Observe that in Example 3.26 we obtained the Bullen type inequality for 3-convex functions.
Obviously, such a result does not hold for 2-convexity. This observation motivates the following conjecture.
Conjecture : Show that for every odd number n the integral $\frac{1}{y-x}\int_a^b f(x)dx$ is less than or equal to the arithmetic mean of the $\frac{n+1}{2}$-points Gauss rule and $\frac{n+3}{2}$-points Lobatto rule.

Remark 5. Note that the integral mean $\frac{1}{y-x}\int_x^y f(t)dt$ occurring in (13) may be written in the form

$$\frac{F(y)-F(x)}{y-x},$$

where F is the antiderivative of f. After this observation, we observe that (13) is an inequality associated with a particular case of the equation

$$\sum_{i=1}^{n}\sum_{p=1}^{N}\gamma_{ip}F_p(a_{ip}x+b_{ip}y)=\sum_{j=1}^{m}\sum_{q=1}^{M}(\alpha_{jq}x+\beta_{jq}y)f_q(c_{jq}x+d_{jq}y) \tag{18}$$

which was considered in [50]. Indeed, it is enough to take $n=1, N=2$ $F_1=F_2=F$ and $\gamma_{11}=1, \gamma_{12}=-1$ to obtain

$$\sum_{i=1}^{n}\sum_{p=1}^{N}\gamma_{ip}F_p(a_{ip}x+b_{ip}y)=F(y)-F(x).$$

On the other hand, taking $m=1$, $\alpha_{1q}=-1, \beta_{1q}=1, q=1,\dots M$; $f_q=a_q f$ and $d_{1q}=1-c_{jq}$ we get

$$\sum_{j=1}^{m}\sum_{q=1}^{M}(\alpha_{jq}x+\beta_{jq}y)f_q(c_{jq}x+d_{jq}y)=(y-x)\sum_{q=1}^{M}a_q f(c_{1q}x+(1-c_{1q})y).$$

Thus, after such substitutions, we arrived at the equation connected directly with (13).

Therefore, it seems of interest to investigate in future research inequalities connected with other particular cases of (5).

4 QUASI-ARITHMETIC MEANS.([42])

In this section, we present the Sugeno Integral of Hermite-Hadamard Inequality for the case of quasi-arithmetically convex (q-ac) functions which acts as a generator for all quasi-arithmetic means in the frame work of Sugeno Integral.
The following theorem gives the Sugeno integral variant of the inequality (1).

Theorem 4.1 ([27])**.** *let $f : [a, b] \subset (0, \infty) \longrightarrow [0, +\infty)$ be a convex function with $f(a) \neq f(b)$. Then*

$$(S)\int_a^b f d\mu \leqslant \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a, b] \cap \left\{x \geqslant \frac{\alpha(b-a) + af(a) - bf(b)}{f(b) - f(a)}\right\}\right)\right)$$

where
$\Gamma = [f(a) \wedge f(b), f(a) \vee f(b)]$.

In the present work we formulate results connected to the Hermite-Hadamard inequality for quasi-arithmetic means and its Sugeno integral variant. This yields a generator for all the quasi-arithmetic means, for example see results [27].

Now let us recall some classical results on quasi-arithmetic means and Hermite-Hadamard inequalities for them.

Definition 4.1. Let $I \subset \mathbb{R}$ be an interval, and let $\phi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ be strictly monotone and continuous functions. We say that $f : I \to \mathbb{R}$, is (ϕ, ψ)-quasi-arithmetically convex ((ϕ, ψ)-q-ac) if $f(I) \subset J$ and the following inequality

$$f(M_\phi(x, y, t)) \leqslant M_\psi(f(x), f(y), t) \tag{19}$$

holds for all $x, y \in I, t \in [0, 1]$. Here $M_\phi : I \times I \times [0, 1] \to I$ defined by

$$M_\phi(x, y, t) := \phi^{-1}\left((1-t)\phi(x) + t\phi(y)\right). \tag{20}$$

and $M_\psi : J \times J \times [0, 1] \to J$ is defined analogously.

Remark 6. Let us note that we may assume without loss of generality that the generator ϕ is strictly increasing.

Functions $M_\phi : I \times I \times [0, 1] \to I$ and $M_\psi : J \times J \times [0, 1] \to J$ are called quasi-arithmetic means. Let us note that, we have $M_\phi\left(x, y, \frac{1}{2}\right) := \phi^{-1}\left(\frac{\phi(x)+\phi(y)}{2}\right)$. In particular, taking as $\phi : I \to \mathbb{R}$ the identity id_I and as ψ the id_J we obtain

$$M_\phi(x, y, t) = (1-t)x + ty,$$

and

$$M_\psi(a,b,t) = (1-t)a + tb,$$

in other words both are linear means, and (19) becomes the classical convexity.

Using the hyperbolic function $\phi : (0,\infty) \to \mathbb{R}$ given by $\phi(x) = \frac{1}{x}$ in (20), we get the generalized harmonic mean

$$H(x,y,t) := \left((1-t)x^{-1} + ty^{-1}\right)^{-1} = \frac{xy}{tx + (1-t)y}.$$

And using the logarithmic function $\phi : (0,\infty) \to \mathbb{R}$ given by $\phi(x) = \ln x$ in (20), we obtain the generalized geometric mean

$$G(x,y,t) = x^{1-t}y^t.$$

Let us start with the following.

Proposition 2. *[1] Let $I, J \subset \mathbb{R}$ be some intervals, and let $\phi : I \to \mathbb{R}$, and $\psi : J \to \mathbb{R}$, be continuous and strictly monotone functions. Then $f : I \to \mathbb{R}$ is (ϕ,ψ)-qa-c if and only if $g = \psi \circ f \circ \phi^{-1} : \phi(I) \to \mathbb{R}$ is convex.*

Proof. Let M_ϕ, M_ψ be quasi-arithmetic means, generated by ϕ and ψ respectively. Suppose that f is (ϕ,ψ)-qa-c, i.e. for every $x, y \in I$, and $t \in [0,1]$ we have

$$f\left(\phi^{-1}\left((1-t)\phi(x) + t\phi(y)\right)\right) \leqslant \psi^{-1}\left((1-t)\psi(f(x)) + t\psi(f(y))\right). \qquad (21)$$

Now let $g : \phi(I) \to \mathbb{R}$ be defined as $g = \psi \circ f \circ \phi^{-1}$. For arbitrary $a, b \in \phi(I)$, choose $x, y \in I$ so that $a = \phi(x)$, $b = \phi(y)$. Let $t \in [0,1]$ be arbitrary. We get, using (ϕ,ψ)-qa-convexity of f, and taking into account the definition of g,

$$\begin{aligned} &g((1-t)a + tb) = g((1-t)\phi(x) + t\phi(y)) = \\ &\quad \psi\left(f\left[\phi^{-1}((1-t)\phi(x) + t\phi(y))\right]\right) \leqslant (1-t)\psi(f(x)) + t\psi(f(y)) = \\ &\quad (1-t)\psi \circ f(\phi^{-1}(a)) + t\psi \circ f(\phi^{-1}(b)) = (1-t)g(a) + tg(b). \end{aligned}$$

To get the converse, assume that $g = \psi \circ f \circ \phi^{-1}$ is convex, let $x, y \in I$, $t \in [0,1]$ be arbitrary, put $a = \phi(x)$, $b = \phi(y)$. We obtain by convexity of g

$$\begin{aligned} &\psi \circ f\left[\phi^{-1}((1-t)\phi(x) + t\phi(y))\right] = g((1-t)a + tb) \leqslant \\ &\quad (1-t)g(a) + tg(b) = (1-t)\psi \circ f(\phi^{-1}(a)) + t\psi \circ f(\phi^{-1}(b)) = \\ &\quad (1-t)\psi(f(x)) + t\psi(f(y)), \end{aligned}$$

and so the Proposition is proved. □

We can express or compose the Borel probability measure in terms of a derivative and a measure. This gives us a special case of Theorem 1 in [40] as seen in the theorem below.

Theorem 4.2. *[40] Let $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ be intervals, and let $\phi : I \to \mathbb{R}$ be a differentiable and strictly monotone function and let $\psi : J \to \mathbb{R}$ be a strictly monotone and continuous. Further, let $f : I \to \mathbb{R}$ be a (ϕ, ψ)-qa-c function. Then the following inequalities hold*

$$f\left(M_\phi\left(x, y, \frac{1}{2}\right)\right) \leqslant \psi^{-1}\left(\frac{1}{\phi(y) - \phi(x)} \int_x^y (\psi \circ f)(u)\phi'(u)du\right) \leqslant M_\psi\left(f(x), f(y), \frac{1}{2}\right) \quad (22)$$

for all $x, y \in I, x \neq y$.

Proof. Let $g : \phi(I) \to \psi(J)$ be given by $g = \psi \circ f \circ \phi^{-1}$. Since f is (ϕ, ψ)-q-ac, in view of Proposition 2 the function g is convex. Hence, in view of (1) we get

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b g(t)dt \leqslant \frac{g(a) + g(b)}{2}$$

for every $a, b \in \phi(I)$, $a \neq b$. Substituting $a = \phi(x)$ and $b = \phi(y)$ for some $x, y \in I$ we can rewrite the above as

$$\psi\left[f\left(\phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right)\right)\right] \leqslant \frac{1}{\phi(y) - \phi(x)} \int_{\phi(x)}^{\phi(y)} (\psi \circ f)\left(\phi^{-1}(t)\right) dt \leqslant \frac{\psi(f(x)) + \psi(f(y))}{2}.$$

Now, substituting in the integral $u = \phi^{-1}(t)$ or $\phi(u) = t$ we get from the well known theorem on changing variables in the integral

$$\psi\left[f\left(\phi^{-1}\left(\frac{\phi(x) + \phi(y)}{2}\right)\right)\right] \leqslant \frac{1}{\phi(y) - \phi(x)} \int_x^y (\psi \circ f)(u)\, \phi'(u)du \leqslant \frac{\psi(f(x)) + \psi(f(y))}{2}.$$

It remains to take ψ^{-1} on both sides of the above inequalities to get our assertion. □

We have in particular the following

Corollary 4.2.1. *Let $I \subset \mathbb{R}$ be an interval, and let $\phi : I \to \mathbb{R}$ be a differentiable and strictly monotone function. Further, let $f : I \to \mathbb{R}$ be a (ϕ, id)-qa-c function. Then the following inequalities hold*

$$f\left(M_\phi\left(x, y, \frac{1}{2}\right)\right) \leqslant \frac{1}{\phi(y) - \phi(x)} \int_x^y f(u)\phi'(u)du \leqslant \frac{f(x) + f(y)}{2}, \tag{23}$$

for all $x, y \in I, x \neq y$.

Remark 7. We observe that in equation (23), for $\phi(x) = \frac{1}{x}$ we obtain $\phi'(x) = -\frac{1}{x^2}$ and then

$$f\left(\frac{2xy}{x+y}\right) \leqslant \frac{xy}{y-x} \int_x^y \frac{f(t)}{t^2} dt \leqslant \frac{f(x) + f(y)}{2},$$

which is the Hermite-Hadamard inequality for harmonically convex functions see [31]. Similarly for $\phi(x) = \ln x, \phi'(x) = \frac{1}{x}$ we get

$$f(\sqrt{xy}) \leqslant \frac{1}{\ln y - \ln x} \int_x^y \frac{f(t)}{t} dt \leqslant \frac{f(x) + f(y)}{2},$$

which is the Hermite-Hadamard inequality for geometrically convex functions. In the same way we can be able to generate other H-H inequalities from the inequality (23).

Let us consider also the inequality (22). Putting $\phi(x) = \frac{1}{x} = \psi(x)$, we obtain

$$f\left(\frac{2xy}{x+y}\right) \leqslant \frac{y-x}{xy \int_x^y \frac{1}{f(t)t^2} dt} \leqslant \frac{2f(x)f(y)}{f(x) + f(y)},$$

which is the Hermite-Hadamard inequality for functions being harmonic-harmonic convex (HH-convex). Similarly, when we put $\phi = \psi = \ln$, we obtain

$$f(\sqrt{xy}) \leqslant \exp\left(\frac{1}{\ln x - \ln y} \int_x^y \frac{\ln f(t)}{t} dt\right) \leqslant \sqrt{f(x)f(y)},$$

or the Hermite-Hadamard inequality for geometric-geometric convex functions (GG-convex).

We may also mix the two above cases, putting $\phi(x) = \frac{1}{x}$ and $\psi(x) = \ln x$. We would obtain

$$f\left(\frac{2xy}{x+y}\right) \leqslant \exp\left(\frac{xy}{y-x} \int_x^y \frac{\ln f(t)}{t^2} dt\right) \leqslant \sqrt{f(x)f(y)},$$

or the Hermite-Hadamard for geometric-harmonic convex functions (GH-convex).

4.1 Hermite-Hadamard inequality for Sugeno integral based on quasi arithmetically convex functions

In this section we extend the results in theorem 4.2 by its Sugeno counterpart.

Theorem 4.3. *Let $f : \mathbb{R} \to [0, \infty)$ be M_ϕ, M_ψ quasi-arithmetic means, generated by ϕ and ψ respectively. Suppose that f is (ϕ, ψ)-qa-c, i.e. for every $x, y \in I$, and $t \in [0, 1]$ we have*

$$f\left(\phi^{-1}\left((1-t)\phi(x) + t\phi(y)\right)\right) \leqslant \psi^{-1}\left((1-t)\psi(f(x)) + t\psi(f(y))\right).$$

If $f \in \boldsymbol{F}$ then

$$(S)\int_a^b f d\mu \leqslant \begin{cases} \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b] \cap \{x \geqslant Q(\alpha)\}\right)\right), & f(a) < f(b), \\ f(a) \wedge \mu([a,b]), & f(a) = f(b), \\ \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b] \cap \{x \leqslant Q(\alpha)\}\right)\right), & f(a) > f(b), \end{cases} \tag{24}$$

where $\Gamma = \operatorname{conv}\{f(a), f(b)\}$, *and* $Q(\alpha) := \phi^{-1}\left(\frac{\psi(\alpha)(\phi(b)-\phi(a))+\phi(a)\psi(f(b))-\phi(b)\psi(f(a))}{\psi(f(b))-\psi(f(a))}\right)$.

Proof. Let $t_x = \frac{\phi(x)-\phi(a)}{\phi(b)-\phi(a)}$. We have that

$$\phi(x) = (1-t_x)\phi(a) + t_x\phi(b)$$

Thus

$$\begin{aligned} f(x) &= f(\phi^{-1}(\phi(x)) = f(\phi^{-1}((1-t_x)\phi(a) + t_x\phi(b))) \\ &= f(M_\phi(a, b, t_x)) \leqslant M_\psi(f(a), f(b), t_x) \\ &= \psi^{-1}((1-t_x)\psi(f(a)) + t_x\psi(f(b))) =: h(x) \end{aligned}$$

Hence by 6. of Proposition 1 we have

$$(S)\int_a^b f d\mu \leqslant (S)\int_a^b \psi^{-1}((1-t_x)\psi(f(a)) + t_x\psi(f(b))d\mu =: (S)\int_a^b h d\mu$$

So we have

$$(S)\int_a^b f d\mu \leqslant (S)\int_a^b h d\mu = \bigvee_{\alpha\geqslant 0}\left(\alpha \wedge \mu\left([a,b] \cap H_\alpha\right)\right), \tag{25}$$

where $H_\alpha = \{x \in [a, b] : h(x) \geqslant \alpha\}$. Consider the following three cases

(i) $f(a) = f(b)$,

(ii) $f(a) < f(b)$,

(iii) $f(a) > f(b)$.

In the case (i), we get $h = \text{const} = f(a)$. Then $(S)\int_a^b hd\mu = f(a) \wedge \mu([a,b])$.

Let us consider the case (ii). First of all, we notice that $h([a,b]) = [f(a), f(b)]$. So, when we deal with $\alpha \in [0, f(a))$ we get

$$H_\alpha = \{x \in [a,b] : h(x) \geqslant \alpha\} = [a,b],$$

and consequently

$$\mu\left([a,b] \cap H_\alpha\right) = \mu([a,b]),$$

which implies

$$\bigvee_{0 \leqslant \alpha \leqslant f(a)} \left(\alpha \wedge \mu\left([a,b] \cap H_\alpha\right)\right) = f(a) \wedge \mu([a,b])) \,.$$

On the other hand, if $\alpha > f(b)$ then $H_\alpha = \varnothing$, and consequently $\mu\left([a,b] \cap H_\alpha\right) = 0$ which implies

$$\bigvee_{f(b) < \alpha} \left(\alpha \wedge \mu\left([a,b] \cap H_\alpha\right) = 0\right).$$

we now obtain

$$\bigvee_{\alpha \geqslant 0} \left(\alpha \wedge \mu\left([a,b] \cap H_\alpha\right)\right) = \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu\left([a,b] \cap H_\alpha\right)\right),$$

which proves our assertion about Γ in the case (ii), and we can replace (25) with

$$(S)\int_a^b fd\mu \leqslant (S)\int_a^b hd\mu = \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu\left([a,b] \cap H_\alpha\right)\right). \tag{26}$$

Next is to find the formula (24). Assume that $f(a) < f(b)$, the proof in the case $f(a) > f(b)$ is analogous):

$$\begin{aligned}
&h(x) \geqslant \alpha \iff \\
&\quad \psi^{-1}((1-t_x)\psi(f(a)) + t_x\psi(f(b))) \geqslant \alpha \iff \\
&\quad x \geqslant \phi^{-1}\left(\frac{\psi(\alpha)(\phi(b) - \phi(a)) + \phi(a)\psi(f(b)) - \phi(b)\psi(f(a))}{\psi(f(b)) - \psi(f(a))}\right) \iff \\
&\quad x \geqslant Q(\alpha).
\end{aligned}$$

□

Corollary 4.3.1. *Let $f : \mathbb{R} \supset I \to [0,\infty)$ be a qa-c function, satisfying*

$$f\left(\phi^{-1}\left((1-t)\phi(x) + t\phi(y)\right)\right) \leqslant (1-t)f(x) + tf(y)$$

for every $x, y \in I$*, and* $t \in [0,1]$*. If* $f \in \boldsymbol{F}$ *then*

$$(S)\int_a^b f d\mu \leqslant \begin{cases} \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b] \cap \{x \geqslant P(\alpha)\}\right)\right), & f(a) < f(b), \\ f(a) \wedge \mu([a,b]), & f(a) = f(b), \\ \bigvee_{\alpha\in\Gamma}\left(\alpha \wedge \mu\left([a,b] \cap \{x \leqslant P(\alpha)\}\right)\right), & f(a) > f(b), \end{cases} \tag{27}$$

where $\Gamma = \operatorname{conv}\{f(a), f(b)\}$*, and* $P(\alpha) := \phi^{-1}\left(\frac{\alpha(\phi(b)-\phi(a))+\phi(a)f(b)+\phi(b)f(a)}{f(b)-f(a)}\right)$.

Proof. Take $\psi = \mathrm{id}_{[0,\infty)}$ in the previous Theorem. □

Example 4.1. From Theorem 4.3, we can obtain the Hermite-Hadamard inequality for Sugeno integral for different (ϕ, ψ)-qa-means. In particular, considering $\phi(x) = x = \psi(x)$ in (24), we obtain

$$Q(\alpha) = \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}$$

which is the linear mean.
Similarly for $\phi(x) = \frac{1}{x} = \psi(x)$, we have

$$Q(\alpha) = \frac{\alpha ab(f(a) - f(b))}{(a-b)f(b)f(a) + \alpha(bf(a) - af(b))}$$

And this gives us Hermite-Hadamard inequality for Sugeno integral based on harmonic mean.
Next is for $\phi(x) = \ln(x) = \psi(x)$,

$$Q(\alpha) = \exp(R(\alpha))$$

where

$$R(\alpha) = \left(\frac{\ln\alpha(\ln b - \ln a) + \ln a \ln f(b) - \ln b \ln f(a)}{\ln f(b) - \ln f(a)}\right).$$

Thus we obtain Hermite-Hadamard inequality for Sugeno integral based on geometric mean.
We may also consider different means for ϕ and ψ, in particular $\phi(x) = \frac{1}{x}$ and $\psi(x) = \ln x$. We obtain

$$Q(\alpha) = \frac{ab(\ln f(a) - \ln f(b))}{\ln\alpha(a-b) + b\ln f(b) - a\ln f(a)}$$

and this gives Hermite-Hadamard inequality for Sugeno integral based on geometric-harmonic means.
In the same way we can obtain other H-H inequalities for Sugeno integral from (24).

Example 4.2. From Corollary 4.3.1, we can generate the Hermite-Hadamard inequality for Sugeno integral for different qa-means. In particular, putting $\phi(x) = x$ in (27), we obtain

$$P(\alpha) = \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}$$

which is the linear mean.
Similarly for $\phi(x) = \frac{1}{x}$, we have

$$P(\alpha) = \frac{ab\,(f(b) - f(a))}{\alpha(a-b) + bf(b) - af(a)}$$

and this gives us harmonic mean. And for $\phi(x) = \ln(x)$, we obtain geometric mean with

$$P(\alpha) = \exp(Q(\alpha))$$

where

$$Q(\alpha) = \left(\frac{\alpha(\ln a - \ln b) + f(b)\ln a - f(a)\ln b}{f(b) - f(a)}\right).$$

In the same way we can obtain other H-H inequalities for Sugeno integral from (27).

Corollary 4.3.2. *Consider a measure space (X, Σ, μ) where μ is a Lebesgue measure on $X = \mathbb{R}$, then from equation (24) we obtain*

$$(S)\int_a^b f d\mu \leqslant \begin{cases} \bigvee_{\alpha\in\Gamma} (\alpha \wedge (b - Q(\alpha))), & f(a) < f(b), \\ f(a) \wedge b - a, & f(a) = f(b), \\ \bigvee_{\alpha\in\Gamma} (\alpha \wedge (Q(\alpha) - a)), & f(a) > f(b), \end{cases} \tag{28}$$

where

$$Q(\alpha) := \phi^{-1}\left(\frac{\psi(\alpha)(\phi(b) - \phi(a)) + \phi(a)\psi(f(b)) - \phi(b)\psi(f(a))}{\psi(f(b)) - \psi(f(a))}\right).$$

We now compute (28) for particular cases of qa-means. Starting with linear mean where

$$Q(\alpha) = \left(\frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}\right),$$

consider the following cases:

(i) $f(a) = f(b)$,

(ii) $f(a) < f(b)$,

(iii) $f(a) > f(b)$.

Case (i) doesn't change. And for case (ii) the minimum between α and $b - Q(\alpha)$ is attained at their point of intersection due to being strictly monotonic. So,

$$\alpha = b - Q(\alpha) \iff$$

$$\alpha = b - \frac{\alpha(a-b) + af(b) - bf(a)}{f(b) - f(a)} \iff$$
$$\alpha(f(b) - f(a) + a - b) = b(f(b) - f(a)) - af(b) + bf(a).$$

So

$$\alpha = \frac{b(f(b) - f(a)) + bf(a) - af(b)}{f(b) + a - b - f(a)}.$$

Taking into account of Remark 2, and 1. of Proposition 1 we have that,

$$(S)\int_a^b f d\mu \leqslant \frac{b(f(b) - f(a)) + bf(a) - af(b)}{f(b) + a - b - f(a)} \wedge (b-a).$$

The proof for case (iii) is analogous.
Next is the harmonic mean, similarly like in the case of linear mean. We see that

$$\alpha = b - \frac{\alpha ab(f(a) - f(b))}{(a-b)f(a)f(b) + \alpha(bf(a) - af(b))}.$$

So

$$\alpha = \frac{-A \pm \sqrt{B^2 - 4AC}}{2A} \tag{29}$$

where $A = bf(a) - af(b)$, $B = (a-b)f(b)f(a) - b^2 f(a) + abf(b) + abf(a) - abf(b)$ and $C = -b(a-b)f(b)f(a)$ from Remark 2, and 1. of Proposition 1

$$(S)\int_a^b f d\mu \leqslant \alpha \wedge (b-a)$$

α is defined in (29) The proof for case (iii) is analogous. And case (i) is constant. In the same way we can be able to compute (28) for other particular cases of qa-means.

4.2 Conclusion.

In this section, we established the Hermite-Hadamard integral inequality for the Sugeno integral based on quasi-arithmetic means which acts as a generator for all Hermite-Hadamard quasi-arithmetic means in the frame work of Sugeno integral.
In the next section, we present some selected results on Lagrangian means which are not quasi-arithmetic means.

5 Lagrangian mean. ([43]),([55]), ([56])

Some well-known quasi-arithmetic means are Lagrangian. For-example arithmetic and geometric means. However, there some arithmetic means which are not lagrangian like harmonic mean and similarly there some Lagrangian means which are not arithmetic like Logarithmic mean. So in this chapter we present our results about Logarithmic mean.

Definition 5.1. Let f be a continuous, strictly monotonic function, defined on an interval I. We define the Lagrangian mean μ associated with f as

$$\mu(x,y) = \begin{cases} f^{-1}\left(\frac{1}{y-x}\int_x^y f(t)dt\right) & x \neq y \\ \\ x & x = y. \end{cases} \tag{30}$$

for $x, y \in I$.

5.1 Characterization of the logarithmic mean

Definition 5.2. Let I be an interval of real numbers. A function $f : I \longrightarrow (0, \infty)$ is said to be log-convex if $\log(f)$ is convex, or equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1-t)y) \leqslant f(x)^t f(y)^{1-t} \tag{31}$$

We note that if f and g are convex functions and g is monotonic nondecreasing, then $g \circ f$ is convex. Moreover, since $f = \exp(\log(f))$ (here $\log = \ln$), it follows that a log-convex function is convex, but the converse is not true.

Applying (1) for a log-convex function f we obtain the following inequality (see [23, Theorem 2.1] or [20, Theorem 2.5]):

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt \leqslant \mathcal{L}(f(x), f(y)), \tag{32}$$

with the logarithmic mean $\mathcal{L}$ on the right-hand side.
We can go further with generalizations. Let $f\colon I \to \mathbb{R}$ and φ be an increasing function defined on the range of f. The function f is said to be φ-convex whenever $\varphi \circ f$ is convex, that is

$$\varphi(f(tx + (1-t)y)) \leqslant t\varphi(f(x)) + (1-t)\varphi(f(y)), \quad x, y \in I,$$

and if φ is one-to-one,

$$f(tx + (1-t)y) \leqslant \varphi^{-1}(t\varphi(f(x)) + (1-t)\varphi(f(y))), \quad x, y \in I. \tag{33}$$

A special case of φ-convex functions is a class of r-convex functions defined on $(0, \infty)$, with $\varphi(x) = x^r$ for $r \in \mathbb{R}\backslash\{0\}$ and $\varphi(x) = \ln x$ for $r = 0$. For the next result we need a notion of a generalization of logarithmic means, namely so called *extended logarithmic means* defined by

$$L_r(x,y) := \begin{cases} \dfrac{r}{r+1} \cdot \dfrac{y^{r+1} - x^{r+1}}{y^r - x^r}, & r \neq -1, 0,\ x \neq y, \\ xy\dfrac{\ln y - \ln x}{y - x}, & r = -1,\ x \neq y, \\ \dfrac{y - x}{\ln y - \ln x}, & r = 0,\ x \neq y, \\ x, & x = y. \end{cases} \tag{34}$$

Surely, for $r = 0$ we have a logarithmic mean.

The following result from [23] describes the extension of the Hermite-Hadamard inequality for r-convex functions.

Theorem 5.1. *Suppose $f\colon I \to (0, \infty)$ is an r-convex function. Then*

$$\frac{1}{y - x} \int_x^y f(t)dt \leqslant L_r(f(x), f(y)). \tag{35}$$

We show how the right-hand side of the Hermite-Hadamard inequality looks like for a general φ-convex function and when the mean on the right-hand side is a Lagrangian mean like it is in the case of the logarithmic mean in (32).

In what follows we give some short general introduction concerning means. Let $I \subset \mathbb{R}$ be an interval. A function $M\colon I^2 \to \mathbb{R}$ is called a *mean* if

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y), \quad x, y \in I. \tag{36}$$

Every mean is *reflexive*, that is $M(x, x) = x$ for all $x \in I$. A mean M is *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in I$, *homogeneous* if $M(tx, ty) =$

$tM(x,y)$ for $t > 0$ and all $x, y \in I$ such that $tx, ty \in I$, and M is said to be a *strict mean* if the inequalities in (36) are strict whenever $x \neq y$. More information about means can be found for instance in [9].

For $p \in \mathbb{R}$ we define a function $L^{[p]}\colon (0,\infty)^2 \to \mathbb{R}$ by the formula

$$L^{[p]}(x,y) := \begin{cases} \left(\dfrac{y^{p+1} - x^{p+1}}{(p+1)(y-x)}\right)^{1/p}, & p \neq -1, 0,\ x \neq y, \\ \dfrac{y-x}{\ln y - \ln x}, & p = -1,\ x \neq y, \\ \dfrac{1}{e}\left(\dfrac{y^y}{x^x}\right)^{1/(y-x)}, & p = 0,\ x \neq y, \\ x, & x = y, \end{cases} \tag{37}$$

for all $x, y \in (0, \infty)$. The function $L^{[p]}$ is a symmetric, strict, homogeneous mean and it is called the *generalized logarithmic mean* of order p of x and y. As special cases of generalized logarithmic means we obtain the *geometric mean* ($p = -2$), the *logarithmic mean* ($p = -1$), the *identric mean* ($p = 0$), the *arithmetic mean* ($p = 1$). For no p the function $L^{[p]}$ is the harmonic mean. (For more details cf., e.g., [8]).

A common generalization of (34) and (37) is given in the notion of Stolarsky means but we will not deal with such a generalization in the work.

If I is open and $f\colon I \to \mathbb{R}$ is differentiable with one-to-one derivative then by the Lagrange mean value theorem for every $x, y \in I$, $x \neq y$, there exists a uniquely determined $M_f(x,y)$ between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'\left(M_f(x,y)\right).$$

The assumption about f is satisfied by strictly convex, or strictly concave, continuously differentiable functions. The function $M_f\colon I^2 \to I$ defined by

$$M_f(x,y) := \begin{cases} (f')^{-1}\left(\dfrac{f(y) - f(x)}{y - x}\right), & x \neq y \\ x, & x = y \end{cases} \tag{38}$$

is a mean on I and it is called a *Lagrangian mean* (see [4]). These means are reflexive, symmetric and strict. The generalized logarithmic means, $L^{[p]}$, are examples of Lagrangian means (it is enough to take $f(x) = x^{p+1}$ for $p \neq -1, 0$, $f(x) = \log x$ for $p = -1$, and $f(x) = x \log x$ for $p = 0$).

Some properties of Lagrangian means are expressed in the following two facts.

Fact 1. (see [9, Theorem 1, p. 344], [8, Theorems 29], [4, Corollary 7]) *Lagrangian means are equal, $M_f(a,b) = M_g(a,b)$ for all a, b, if and only if for some $\alpha, \beta, \gamma, \alpha \neq 0$,*

$$f(x) = \alpha g(x) + \beta x + \gamma, \quad x \in \mathbb{R}.$$

Fact 2. (see [9, Theorem 1, p. 346], [8, Theorems 30]) *If a Lagrangian mean M_f is homogeneous then for some $p \in \mathbb{R}$, and all a and b we have $M_f(a,b) = L^{[p]}(a,b)$.*

As a consequence of Fact 1 without loss of generality we can assume that f is convex. As a consequence of Fact 2, there is no function f such that M_f is the harmonic mean. Otherwise, since the harmonic mean is homogeneous it would have to be $L^{[p]}$ for some p, and the harmonic mean is not a generalized logarithmic mean.

We proceed now with a theorem which generalizes inequalities (32) or (35).

Theorem 5.2. *Suppose $f\colon I \to \mathbb{R}$. Let φ be a strictly increasing function defined on the range of f and Φ – its primitive function. If f is a φ-convex function, then*

$$\frac{1}{y-x}\int_x^y f(s)ds \leqslant \Lambda_\varphi(f(x), f(y)), \tag{39}$$

where

$$\Lambda_\varphi(x,y) := \begin{cases} \dfrac{y\varphi(y) - \Phi(y) - x\varphi(x) + \Phi(x)}{\varphi(y) - \varphi(x)}, & x \neq y, \\ x, & x = y. \end{cases}$$

Proof. We integrate inequality (33) with respect to t, that is,

$$\int_0^1 f(tx + (1-t)y)dt \leqslant \int_0^1 \varphi^{-1}(t\varphi(f(x)) + (1-t)\varphi(f(y)))dt. \tag{40}$$

Starting with the left-hand side of (40), with the substitution $s := tx + (1-t)y$ we obtain

$$\int_0^1 f(tx + (1-t)y)dt = \frac{1}{x-y}\int_y^x f(s)ds.$$

For the right-hand side of (40), assume first that $f(x) = f(y)$ in order to get $f(x)$ as a result of integration. Suppose now $f(x) \neq f(y)$. We shall use the so called Laisant formula, i.e.,

$$\int \varphi^{-1}(z)dz = z\varphi^{-1}(z) - \Phi(\varphi^{-1}(z)) + C,$$

where Φ is a primitive function of φ and C is an arbitrary constant.

With $z := t\varphi(f(x)) + (1-t)\varphi(f(y))$ we have

$$\begin{aligned}
&\int_0^1 \varphi^{-1}(t\varphi(f(x)) + (1-t)\varphi(f(y)))dt \\
&= \frac{1}{\varphi(f(x)) - \varphi(f(y))} \int_{\varphi(f(y))}^{\varphi(f(x))} \varphi^{-1}(z)dz \\
&= \frac{1}{\varphi(f(x)) - \varphi(f(y))} \Big[z\varphi^{-1}(z) - \Phi(\varphi^{-1}(z)) \Big]_{\varphi(f(y))}^{\varphi(f(x))} \\
&= \frac{f(x)\varphi(f(x)) - \Phi(f(x)) - f(y)\varphi(f(y)) + \Phi(f(y))}{\varphi(f(x)) - \varphi(f(y))}.
\end{aligned}$$

This completes the proof. □

Since the logarithmic mean is a special case of Lagrangian means, we generalize the log-convexity in equation (31) to see if the inequality in equation (32) satisfied by log-convex functions can be generalized for Lagrangian means other than logarithmic one. In the next theorem, we consider the right hand side of inequality (39) to be the general form of the Lagrangian mean, generated by φ and solve the corresponding equation.

Theorem 5.3. *Let φ be a strictly increasing real function from the class $\mathcal{C}^3$, defined on an interval J, and with non-vanishing first and second derivatives. Then*

$$\Lambda_\varphi(x, y) = M_\varphi(x, y),$$

for all $x, y \in J$, if and only if $\varphi(x) = \frac{1}{a}\ln|ax+b|+c$ for some $a, b, c \in \mathbb{R}$, $a \neq 0$, such that $ax + b$ for all $x \in J$ has a constant sign.

Proof. It is easy to check that $\varphi = \ln$ is a solution of equation (5.3).

Solving (5.3) we immediately assume that $x \neq y$. We have

$$\frac{x\varphi(x) - \Phi(x) - y\varphi(y) + \Phi(y)}{\varphi(x) - \varphi(y)} = (\varphi')^{-1}\left(\frac{\varphi(x) - \varphi(y)}{x - y}\right). \tag{41}$$

For solving (41), we denote

$$\Lambda(x, y) := \frac{x\varphi(x) - \Phi(x) - y\varphi(y) + \Phi(y)}{\varphi(x) - \varphi(y)}$$

and

$$M(x, y) := (\varphi')^{-1}\left(\frac{\varphi(x) - \varphi(y)}{x - y}\right). \tag{42}$$

Since

$$\varphi'(M(x,y)) = \frac{\varphi(x)-\varphi(y)}{x-y},$$

then

$$\varphi''(M(x,y)) \cdot \frac{\partial M}{\partial x}(x,y) = \frac{\varphi'(x)(x-y)-\varphi(x)+\varphi(y)}{(x-y)^2}$$

$$\varphi'''(M(x,y)) \cdot \left[\frac{\partial M}{\partial x}(x,y)\right]^2 + \varphi''(M(x,y)) \cdot \frac{\partial^2 M}{\partial x^2}(x,y)$$
$$= \frac{\varphi''(x)(x-y)^2 - 2[\varphi'(x)(x-y)-\varphi(x)+\varphi(y)]}{(x-y)^3}.$$

In what follows we observe that there exists the limit of $\frac{\partial^2 M}{\partial x^2}(x,y)$ as y tends to x and we compute it. For, first we compute

$$\lim_{y\to x} \frac{\partial M}{\partial x}(x,y) = \frac{1}{\varphi''(x)} \cdot \lim_{y\to x} \frac{\varphi'(x)(x-y)-\varphi(x)+\varphi(y)}{(x-y)^2} = \frac{1}{2}$$

$$\lim_{y\to x} \frac{\varphi'(x)(x-y)-\varphi(x)+\varphi(y)}{(x-y)^2} = \lim_{y\to x} \frac{-\varphi'(x)+\varphi'(y)}{-2(x-y)} = \frac{\varphi''(x)}{2}$$

$$\lim_{y\to x} \frac{\varphi''(x)(x-y)^2 - 2[\varphi'(x)(x-y)-\varphi(x)+\varphi(y)]}{(x-y)^3} = \frac{\varphi'''(x)}{3}.$$

Therefore, there exists the limit of $\frac{\partial^2 M}{\partial x^2}(x,y)$ the limit as $y \to x$,

$$\frac{1}{4}\varphi'''(x) + \varphi''(x) \cdot \left(\lim_{y\to x} \frac{\partial^2 M}{\partial x^2}(x,y)\right) = \frac{\varphi'''(x)}{3}$$

and

$$\lim_{y\to x} \frac{\partial^2 M}{\partial x^2}(x,y) = \frac{1}{12}\frac{\varphi'''(x)}{\varphi''(x)}. \tag{43}$$

Next we compute the partial derivatives of Λ with respect to x:

$$\frac{\partial \Lambda}{\partial x}(x,y) = \varphi'(x) \cdot \frac{\varphi(y)(y-x)+\Phi(x)-\Phi(y)}{[\varphi(x)-\varphi(y)]^2}$$

$$\frac{\partial^2 \Lambda}{\partial x^2}(x,y) = \varphi''(x) \cdot \frac{\varphi(y)(y-x)+\Phi(x)-\Phi(y)}{[\varphi(x)-\varphi(y)]^2}$$
$$+ \varphi'(x) \cdot \frac{[\varphi(x)-\varphi(y)]^2 - 2\varphi'(x)[\varphi(y)(y-x)+\Phi(x)-\Phi(y)]}{[\varphi(x)-\varphi(y)]^3}.$$

In order to compute the limit of $\frac{\partial^2\Lambda}{\partial x^2}(x,y)$ as $y \to x$, first we compute the following limits:

$$\lim_{y\to x} \frac{\varphi(y)(y-x)+\Phi(x)-\Phi(y)}{[\varphi(x)-\varphi(y)]^2} = \frac{1}{2\varphi'(x)},$$
$$\lim_{y\to x} \frac{[\varphi(x)-\varphi(y)]^2-2\varphi'(x)[\varphi(y)(y-x)+\Phi(x)-\Phi(y)]}{[\varphi(x)-\varphi(y)]^3} = -\frac{\varphi''(x)}{3[\varphi'(x)]^2}.$$

Therefore,

$$\lim_{y\to x} \frac{\partial^2\Lambda(x,y)}{\partial x^2} = \varphi''(x)\cdot\frac{1}{2\varphi'(x)} - \varphi'(x)\cdot\frac{\varphi''(x)}{3[\varphi'(x)]^2} = \frac{\varphi''(x)}{6\varphi'(x)}. \tag{44}$$

From equations (43) and (44) we derive that the solution φ of equation (41) satisfies

$$\frac{1}{12}\frac{\varphi'''(x)}{\varphi''(x)} = \frac{1}{6}\frac{\varphi''(x)}{\varphi'(x)}. \tag{45}$$

That is,

$$\ln|\varphi''(x)| = 2\ln|\varphi'(x)| + \ln C,$$

with a positive C, and

$$\varphi''(x) = -a[\varphi'(x)]^2$$

with a nonzero constant a. Further, since $\varphi'(x)$ does not vanish, we have

$$\frac{\varphi''(x)}{[\varphi'(x)]^2} = -a$$

and

$$\varphi(x) = \frac{1}{a}\ln|ax+b| + c$$

with $a, b, c \in \mathbb{R}$, $a \neq 0$. □

Corollary 5.3.1. *Suppose $f\colon I \to (0,\infty)$. Let $\varphi\colon (0,\infty) \to \mathbb{R}$ be a strictly increasing function from the class $\mathcal{C}^3$ and with non-vanishing first and second derivatives. If $\Lambda_\varphi(x,y) = M_\varphi(x,y)$, for all $x,y \in I$, then the logarithmic mean is the only one (up to a translation) which is both Lagrangian and of the form Λ_φ. More exactly,*

$$\Lambda_\varphi(x,y) = M_\varphi(x,y) = \frac{y-x}{\ln(ay+b)-\ln(ax+b)} - \frac{b}{a}$$

for some $a > 0$, $b \geqslant 0$.

Therefore, logarithmic mean is the only (up to a affine translation) Lagrangian mean satisfying the right-hand side of the Hermite-Hadamard inequality.

5.2 A note on a result by J. Sándor

The note concerns a result contained in [55]. The author claims, among others, that the following theorem holds (cf. Theorem 2.5 in [55]).

Theorem 5.4. *Let $f : [a,b] \subset (0,\infty) \longrightarrow (0,\infty)$ be a geometric convex function such that the application $x \longrightarrow \frac{f(x)}{x}$ is increasing. Then one has the inequalities*

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \leqslant \frac{1}{A(a,b)}L(af(a),bf(b)) \leqslant L(f(a),f(b)). \qquad (46)$$

Here A stands for the arithmetic mean, and L for the logarithmic mean, i.e. the mean defined by

$$L(x,y) := \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y. \end{cases}$$

It is observed that the last inequality on the right hand side does not hold. Indeed,consider the following example.

Example 5.1. Let $[a,b] = [1,2]$ and let $f : [1,2] \longrightarrow (0,\infty)$ be given by $f(x) = x^2$. Then f is geometric convex (even geometric affine), since

$$\bigwedge_{x,y\in[1,2]} \bigwedge_{\lambda\in[0,1]} \left(f(x^\lambda y^{1-\lambda}) = (x^\lambda y^{1-\lambda})^2 = (x^\lambda)^2 (y^{1-\lambda})^2 = f(x^\lambda) f(y^{1-\lambda}) \right).$$

Also the application $x \longrightarrow \frac{f(x)}{x} = x$ is increasing. However, if we take $a = 1$ and $b = 2$ and substitute it into (46) we get

$$\frac{1}{A(a,b)}L(af(a),bf(b)) = \frac{2}{3}\frac{7}{3\ln 2} = \frac{14}{9\ln 2} > \frac{3}{2\ln 2} = L(f(a),f(b)).$$

When studying the proof of Theorem 5.4, we see that while the first inequality is a consequences of previous results, the last inequality is a consequence of the following lemma (cf. Lemma2.3 in [55]).

Lemma 5.5. *If $\frac{q}{p} \geqslant \frac{b}{a} \geqslant 1$, then*

$$L(pa,qb) \leqslant L(p,q)A(a,b). \qquad (47)$$

The proof is based on the study of a function k defined for $u \geqslant v$ by

$$k(u) := (v-1)(u+1)\ln(uv) - 2(uv-1)\ln v.$$

The author writes that k is increasing, and hence the inequality (47) follows. The point is that the function k does not correspond to the inequality (47). Indeed, we have

Lemma 5.6. *If* $\frac{q}{p} \geqslant \frac{b}{a} \geqslant 1$, *then*

$$\begin{cases} L(pa, qb) > L(p,q)A(a,b), & \text{if } \frac{q}{p} > \frac{b}{a} > 1, \\ L(pa, qb) = L(p,q)A(a,b), & \text{if } \frac{b}{a} = 1 \text{ or } \frac{b}{a} = \frac{q}{p}. \end{cases} \tag{48}$$

Proof. Let $u = \frac{q}{p}$ and $v = \frac{b}{a}$. By our assumptions we may rewrite the inequality (48) as (we admit that $\frac{u-1}{\ln u} = 1$, if $u = 1$)

$$\begin{cases} \frac{uv-1}{\ln(uv)} > \frac{v+1}{2}\frac{u-1}{\ln u}, & \text{if } u > v > 1, \\ \frac{uv-1}{\ln(uv)} = \frac{v+1}{2}\frac{u-1}{\ln u}, & \text{if } v = 1 \text{ or } v = u. \end{cases} \tag{49}$$

We see that (49) is equivalent to

$$\begin{cases} 2(uv-1)\ln(u) - (v+1)(u-1)\ln(uv) > 0, & \text{if } u > v > 1, \\ 2(uv-1)\ln(u) - (v+1)(u-1)\ln(uv) = 0, & \text{if } v = 1 \text{ or } v = u. \end{cases} \tag{50}$$

Fix a $v \geqslant 1$ and define the application $k_v : [v, \infty) \longrightarrow \mathbb{R}$ given by

$$k_v(u) = 2(uv-1)\ln(u) - (v+1)(u-1)\ln(uv). \tag{51}$$

Let us note that $k_v(v) = 0$. Let us also observe that $k_1 = 0$, which amounts to the second part of (50), and a fortiori (49) or (48). Thus to prove (50), and hence (49), and consequently (48) it is enough to assume that $v > 1$, and show that k_v is strictly increasing. Let us calculate the derivative of k_v.

We have for each $u \in (v, \infty)$

$$\begin{aligned} &k_v'(u) = \\ &2\left[v\ln(u) + (uv-1)\frac{1}{u}\right] - (v+1)\left[\ln(uv) + (u-1)\frac{1}{u}\right] = \\ &2v\ln(u) + 2v - \frac{2}{u} - (v+1)\left[\ln(u) + 1 - \frac{1}{u}\right] - (v+1)\ln(v) = \\ &(2v - v - 1)\left[\ln(u) + 1 - \frac{1}{u}\right] - (v+1)\ln(v) + (v-1)\frac{2}{u} = \\ &(v-1)\left[\ln(u) + 1 + \frac{1}{u}\right] - (v+1)\ln(v). \end{aligned} \tag{52}$$

Put $h(u) := \ln(u) + 1 + \frac{1}{u}$, $u \geqslant 1$. Since

$$h'(u) = \frac{1}{u} - \frac{1}{u^2} = \frac{u-1}{u^2} > 0,\ u > 1,$$

we see that h is increasing in $[1, \infty)$. Taking into account (52), we see that

$$k_v'(u) > (v-1)h(v) - (v+1)\ln v,\ u \geqslant v. \tag{53}$$

Denote by φ the mapping defined on $[1, \infty)$ by the formula

$$\varphi(v) = (v-1)h(v) - (v+1)\ln v = -\ln v^2 + \frac{v^2-1}{v}.$$

We have $\varphi(1) = 0$, and $\varphi'(v) = -2\frac{1}{v} + 1 + \frac{1}{v^2} = \left(1 - \frac{1}{v}\right)^2 > 0$, $v > 1$. Hence φ is increasing, and consequently $\varphi(v) > 0$, $v > 1$. HenceThus, in view of (53), we get $k_v(u) > 0$ for all $u > v$. □

6 Application of Fuzzy Integral in Portfolio Risk Management. ([45])

6.1 Introduction

Portfolio risk management is a critical aspect of asset allocation and investment management that focuses on identifying, assessing, and mitigating risks associated with a portfolio of assets.
A portfolio, comprising investment classes such as stocks, bonds with different ratings, equity,commodities, and real estate, is subject to numerous risks arising from market fluctuations (like movements in stock prices, interest rates, exchange rates etc.), economic conditions and other factors.
Portfolio risk management plays a pivotal role in the investment decision-making process because it involves analyzing the risk-return characteristics of individual assets and their interactions within a portfolio to construct an optimal allocation that aligns with investors' risk tolerance and objectives. By managing risks effectively, investors can protect their capital from potential losses and enhance the potential for achieving desired returns.

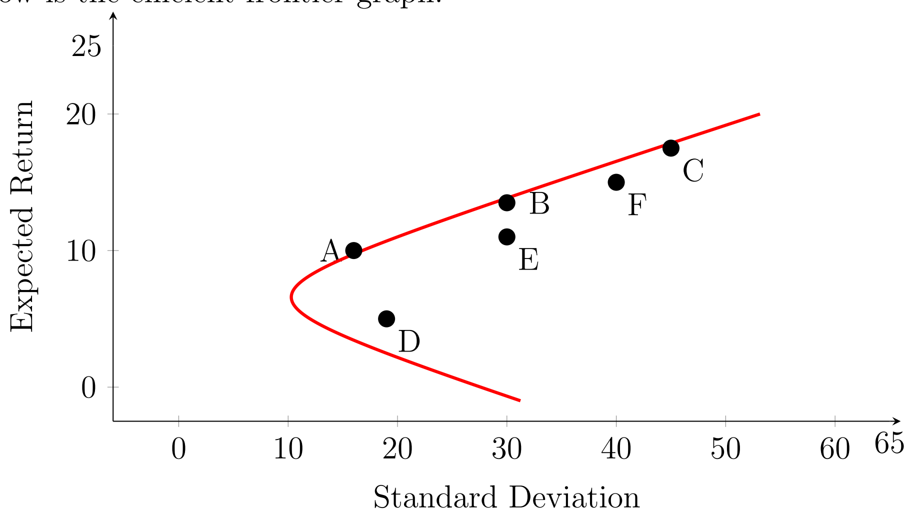
In the next subsection we explore the belief overview of the fundamental principles of Modern Portfolio theory and its relevance in portfolio risk management.

6.1.1 Modern Portfolio Theory

Modern Portfolio Theory (MPT), introduced by Harry Markowitz in 1952, is one of the most important and influential economic theories dealing with finance and investment. It revolutionized the field of investment management by providing a systematic framework for portfolio construction and risk management. It is an investment theory based on the idea that risk-averse investors can construct portfolios to maximize the expected return for a given level of risk. It aims at optimize the risk-return trade-off by diversifying investments across different assets. Diversification is away of holding a combination of investments across a mix of different assets with low or negative correlations and this minimizes the impact of individual asset volatility on the overall portfolio. By doing so, investors can maximize higher returns with lower risk. MPT is based on the fact that investors are rational decision-makers who seek to maximize returns while minimizing risk. Risk is measured by the standard deviation of an asset's returns, and return refers to the potential gain or loss an investment can generate. We can represents a set of portfolios that offer the highest expected returns for a given level of risk or the lowest risk for a given level of expected return on an efficient frontier [35],[36],[37].

6.1.2 Efficient frontier

The efficient frontier is a powerful concept in Modern Portfolio Theory that guides investors in constructing diversified portfolios that maximize expected returns for a given level of risk or minimize risk for a given level of expected returns. The efficient frontier helps investors make informed decisions about their asset allocation because by choosing a portfolio on the efficient frontier that aligns with their risk tolerance and financial goals, investors can achieve the best trade-off between risk and return. It's derived by combining different assets with varying risk and return characteristics to achieve an optimal portfolio allocation.
The efficient frontier is created by plotting all possible portfolios of risky assets on a graph with expected return on the y-axis and risk (as measured by standard deviation) on the x-axis. All points on the efficient frontier represent optimal portfolios with the highest expected return for a given level of risk or the lowest risk for a given level of expected return and portfolios lying below the efficient frontier are sub-optimal because they provide lower returns for the same level of risk or higher risk for the same level of return.
Below is the efficient frontier graph.



As you can see, the efficient frontier is a curve that slopes upward from the lower left corner of the graph to the upper right corner. There are three portfolios on the efficient frontier, labeled A, B, and C.

1. Portfolio A is located on the left side of the efficient frontier. This means

that it has a lower expected return than Portfolio B or Portfolio C, but it also has a lower level of risk.

2. Portfolio B is located in the middle of the efficient frontier. This means that it has an expected return that is higher than Portfolio A, but it also has a higher level of risk.

3. Portfolio C is located on the right side of the efficient frontier. This means that it has the highest expected return of the three portfolios, but it also has the highest level of risk.

So an investor who is risk-averse might choose Portfolio A. This portfolio would have a lower expected return than Portfolio B or Portfolio C, but it would also have a lower level of risk. And an investor who is risk-seeking might choose Portfolio C. This portfolio would have the highest expected return of the three portfolios, but it would also have the highest level of risk. Ultimately, the choice of which portfolio to invest in is a personal decision that should be made based on an investor's individual risk tolerance and return objectives.

6.1.3 Criticism of Modern Portfolio theory and why we need a non-additive measure

The traditional models and analysis procedures for portfolio optimization are, in most cases, based on the assumption that the distribution of returns of an asset is normal. This means that in practice, a portfolio of stocks undergoes small percentage daily losses and gains much more often than negligible or extreme fluctuations.
While as the MPT theorem appears to be a good method of optimal portfolio construction and management, it uses the mathematical concept of variance to quantify risk. And this can only be justified under the assumption of elliptically distributed returns such as normal distribution returns.
Modern portfolio theory has been criticized because it assumes that returns follow a Gaussian distribution(normal distribution).
In 1960s, Benoit Mandeltbrot and Eugene Fama showed the inadequacy of this assumption and proposed the use of a stable distribution instead. Stefan Mittnik and Svetlozar Rachev presented strategies for deriving optimal portfolios in such settings [59],[60],[63].
B. Mandelbrot, a French mathematician of Polish origin who researches in the field of fractal geometry, disagreed with the applicability of the Gaussian distribution in explaining the reality of financial markets. And he suggested

that extreme movements are much more likely than the commonly used models in finance predict. This is why traditional methods in risk management and finance are being increasingly criticized.
Traditional risk measures such as standard deviation or Value at Risk (VaR) often rely on precise probabilities or distributions, which may not accurately capture the complex and unpredictable nature of financial markets. However fuzzy measures, through the incorporation of haircuts, captures the imprecise and uncertain nature of risk and this offer a more comprehensive approach to risk analysis, enabling us to make better-informed investment .
The assumption of a normal distribution underestimates the probability of large and important price movements for portfolio optimization. And relying on the correlation matrices, it fails to capture the relevant dependence structure among the characteristics of assets. So we propose a new non-additive(fuzzy) aggregation function which not only doesn't assume any distribution but it captures the diversification and dependencies or complexities with in the characteristics of an asset . So by employing non-additive fuzzy measures in portfolio risk management, we gain a more accurate understanding of the portfolio's risk profile and can make informed investment decisions accordingly.

6.2 Optimization approach to find λ

In situations where the domain is extensive, solving the polynomial equation to determine the optimal value of λ can become intricate and convoluted. To address this, we propose an iterative approach as an alternative, allowing us to obtain the optimal value for λ without having to solve complex polynomial functions. This iterative method proves advantageous, particularly in real-world scenarios characterized by larger domains, surpassing the limitations of conventional techniques and that's the gradient descent.

6.2.1 Gradient Descent

Gradient descent is an iterative optimization algorithm for finding the minimum of a function. It is a very powerful algorithm that is used in a wide variety of applications, including machine learning, data science, optimization, finance, and engineering. It aims to adjust the parameters of a model iteratively by taking steps proportional to the negative gradient of the cost function at a given point.
Gradient descent works by starting at a random point and then repeatedly

moving in the direction of the negative gradient of the function. The gradient of a function is a vector that points in the direction of the steepest ascent of the function. So, by moving in the direction of the negative gradient, we are moving in the direction of the steepest descent

The mathematical expression for gradient descent is as follows:

$$x_t = x_{t-1} - \alpha \nabla f(x_{t-1}),\ t \in \mathbb{N}$$

where:
x_t is the value of the parameter at iteration t,
x_{t-1} is the value of the parameter at iteration $t-1$,
α is the learning rate and,
$\nabla f(x_{t-1})$ is the gradient of the function at x_{t-1}
The main parameters of gradient descent are the learning rate and the number of iterations. The learning rate controls how much the parameter is updated at each iteration. A larger learning rate will cause the parameter to be updated more quickly. However, the algorithm may overshoot the minimum and fail to converge. A smaller learning rate will cause the parameter to be updated more slowly, but it may also cause the algorithm to converge more slowly. The number of iterations controls how many times the algorithm will update the parameter. A larger number of iterations will increase the accuracy of the solution, but it will also take longer to find the solution.
Besides the learning rate, Initial Parameters (t_0) the initial values of the parameters influence the starting point of the optimization process. Different initial values can lead to different local minima or convergence rates.

6.2.2 Cost Function

The cost function quantifies the error or discrepancy between the predicted output of a model and the actual output. It represents the objective to be minimized during training. The choice of the cost function depends on the specific problem and model.
The cost function should be differentiable since the gradient descent algorithm relies on calculating the gradient $\nabla f(x_{t-1})$ to update the parameters. Differentiability ensures that the algorithm can find the direction of steepest descent to iteratively approach the minimum.
With the form provided in Definition 2.6, we proceed to introduce our optimization methodology. We convert equation (7) into a cost function, or penalty function, and employ an iterative technique, specifically gradient de-

scent, to determine the optimal value of λ that minimizes the cost function. Based on equation (7), we formulate the following cost function.

Definition 6.1. Let $X_i, i \in \{1, \cdots, n\}$ be pairwise disjoint sets and let X represent the union of all these X_i sets. Let $m_\lambda : \mathcal{X} \to [0,1]$, the cost function $C : (-1, \infty) \to \mathbb{R}_+$, is defined as

$$C(\lambda) = \frac{1}{2}\left(1 - \sum_{j=1}^{n} \sum_{\hat{X} \in \sigma(j)} \lambda^{j-1} m_\lambda(\hat{X})\right)^2 \tag{54}$$

and $m_\lambda(\hat{X}) = \prod_{\hat{X}_* \in \hat{X}} m_\lambda(\hat{X}_*)$.

6.2.3 Algorithm for Gradient Descent

Next, we use gradient descent to find the best value of λ such that equation (54) is approximately zero.
Implementing the gradient descent update rule. Below is the algorithm for the Gradient Descent optimization method

1. Initialize the weight (parameter) λ to some initial value.
2. Set the learning rate α (step size) to a small positive value.
3. Iterative Update: Repeat the following steps until the convergence criterion is met:
   a. Compute the gradient of the cost function with respect to λ i.e.
   $\nabla C(\lambda) = \left(\sum_{j=1}^{n} \sum_{\hat{X} \in \sigma(j)} \lambda^{j-1} m_\lambda(\hat{X}) - 1\right)\left(\sum_{j=2}^{n} \sum_{\hat{X} \in \sigma(j)} (j-1)\lambda^{j-2} m_\lambda(\hat{X})\right)$.
   where $m_\lambda(\hat{X}) = \prod_{\hat{X}_* \in \hat{X}} m_\lambda(\hat{X}_*)$.
   b. Update λ at iteration t using the gradient at iteration $t-1$ and learning rate:
   $\lambda_t := \lambda_{t-1} - \alpha \nabla C(\lambda_{t-1})$, $t \in \mathbb{N}$.
   c. Repeat steps a and b for a predefined number of iterations or until the cost function reaches a desired threshold.
4. Return the optimized weights λ
   .

6.2.4 Computer Code for Computing m_λ-measure

6.2.5 Function to Compute λ

import numpy as np
import itertools as it

```
def compute_lambda(lad, X, iteration, error, learning_rate):
    lamda = lad
    power = []
    if len(X) > 4:
        a = learning_rate
    else:
        a = 1
    for i in range(1, len(X) + 1):
        list_combinations = it.combinations(X, i)
        for item in list(list_combinations):
            power.append(item)
    co = [0]
    for j in range(iteration):
        cs = 1
        cs1 = 0
        for i in power:
            cs = cs - np.float64(np.power(lamda, len(i) - 1)
            * np.prod(i))
            if len(i) > 1:
                cs1 = cs1 + np.float64((len(i) - 1)
                * ((lamda)**(len(i) - 2)) * np.prod(i))
        cost = (cs**2) * 0.5
        co.append(cost)
        lamda = lamda + a * (cs * cs1)
        if co[-1] <= error:
            print("lambda:", round(lamda, 4))
            break
    if co[-1] > error:
        print('cost greater than accepted error')
    return round(lamda, 4)
```

6.2.6 Function to Compute Pairs

```
def pairs(lad, X):
    new_list=[]
```

```
for i in range(1,len(X)+1):
    new_list.append("X_"+str(i))
if len(X)> 2:
    lamda=lad
    power=[]
    pairs=[]
    paip=[]
    final=[]
    all_set=[]
    powers_new=[]
    new_list_pairs=[]
    for i in range(1,len(X)+1):
        list_combinations =it.combinations(X, i)
        for item in list(list_combinations):
            power.append(item)
    for i in range(1,len(new_list)+1):
        list_combinations =it.combinations(new_list, i)
        for item in list(list_combinations):
            powers_new.append(item)
    for i in power:
        if len(i)!=1 and len(i) != len(X):
            pairs.append(i)
    for i in powers_new:
        if len(i)!=1 and len(i) != len(new_list):
            new_list_pairs.append(i)
    npower=[]
    for i in pairs:
        for j in range(1,len(i)+1):
            list_combinations1 =it.combinations(i, j)
            for item in list(list_combinations1):
                npower.append(item)
                ncs=0
                for ii in npower:
                    ncs= ncs+ (((lamda)**(len(ii)-1))*np.prod(ii))
        paip.append(ncs)
    final.append(round(paip[0],4))
    for i in range(1,len(paip)):
        final.append(round(paip[i]-paip[i-1],4))
    for i in range(1,len(X)+1):
        all_set.append("X_"+str(i))
        print("m-lambda([X_",i,"])=",X[i-1], sep="")
```

```
        for i in range(len(pairs)):
            print("m-lambda([" + 'U'.join(new_list_pairs[i]) + ']' ")=
            final[i],sep="")
        print("m-lambda([" + 'U'.join(all_set) + ']' ")=",1, sep="")
    elif len(X)==2:
        for i in range(1,len(X)+1):
            print("m-lambda([X_",i,"])=",X[i-1], sep="")
        print("m-lambda([X_1UX_2])=",1, sep="")
```

Function to Compute m_λ-measure

We combine the above python functions to get the m_λ-measure function

```
def m_lambda_measure(X):
    lad=0
    iteration=1000000
    error=0.000000000001
    learning_rate=0.0001
    ladi=compute_lambda(lad,X,iteration,error,learning_rate)
    return pairs(ladi,X)
```

So upto this end we've an overview of fuzzy measure theory, as a mathematical framework used to model and quantify uncertainty and imprecision. It provides a more nuanced and flexible approach to risk assessment compared to traditional deterministic methods. Since fuzzy sets and membership functions, enables the representation and manipulation of imprecise information. By employing fuzzy measure theory, we can capture the inherently uncertain nature of risk within portfolio management. So building on the foundation of fuzzy measure theory, and taking advantage of non-additive measure(fuzzy) in portfolio risk management.
In order to capture the inherent uncertainties associated with the market value of assets, we employ the concept of haircuts to portray the fuzzy densities. So, next we explore the concept of haircuts.

6.3 Haircuts

In the context of portfolio risk management, haircuts represent a percentage reduction in the market value of an asset. The purpose of applying haircuts is to protect against potential losses if the asset's value declines. By using haircuts as an alternative risk measure of uncertainties(fuzzy densities), we can capture and quantify the imprecise and uncertain aspects of asset risk. This enables us to incorporate more comprehensive risk assessment techniques within MPT, enhancing our ability to construct portfolios that account for a range of potential outcomes.

Different assets or asset classes are assigned specific haircuts based on their risk characteristics. Factors such as market fluctuations, liquidity, credit quality, and volatility are taken into consideration when determining the magnitude of haircuts for each asset. By adjusting asset values using haircuts, we can better reflect the uncertain and imprecise nature of risk, allowing for a more accurate representation of asset values within the portfolio.

The haircut can be thought of as a measure of the uncertainty associated with the asset's value. For example, an asset with a high haircut would be considered riskier than an asset with a lower haircut.

Example 6.1. Let's examine assets X and Y. Asset X is valued at 100 Polish Zloty (PLN) with a 10% haircut, while Asset Y is valued at 100 Polish Zloty (PLN) with a 20% haircut. From this, we can deduce that asset Y is riskier than asset X.

Through the combination of fuzzy integral and fuzzy measures represented by haircuts, we derive an all-encompassing risk score for the portfolio. This thorough risk assessment considers asset weights and the distribution of haircuts among various assets.

To illustrate the practical implementation of these concepts in portfolio construction, we provide a compelling case study. By integrating haircuts as fuzzy measures in the portfolio risk assessment, we demonstrate how this approach enhances our comprehension of the portfolio's risk profile. We present the case study results, highlighting the valuable insights gained from this methodology.

6.4 Application

Modern Portfolio Theory (MPT) revolutionized portfolio construction by emphasizing the importance of diversification to optimize risk and return. Although various aggregation functions, such as the Sugeno and Choquet integrals, have been widely applied in decision-making and risk assessment, they fail to explicitly consider diversification as illustrated in the next example below.
So we propose the development of a new aggregation function that accounts for the crucial aspect of diversification. And by introducing this new function, we aim to enhance the risk management capabilities within Modern Portfolio Theory (MPT) and further refine portfolio optimization strategies which enables portfolio managers and investors to accurately evaluate and optimize the risk-return trade-off.

Let's begin by the definition and an example on the d-Choquet integral(aggregation function).

Definition 6.2. (see[10, Definition 3.1])
Let μ be a fuzzy normalized measure defined on the set of sets $X = \bigcup_{i=1}^{n} X_i$, where X_i are pairwise disjoint sets. The integral of a function $f : X \to [0, 1]$ with respect to the fuzzy measure μ is given by:

$$(O)\int_A f d\mu = \sum_{i=1}^{n}[f(X_i) - f(X_{i-1})]^2 \mu(A_i), \tag{55}$$

Here, $\mu(A_1) = \mu(X_1 \cup X_2 \cup \ldots \cup X_n)$, $\mu(A_2) = \mu(X_2 \cup X_3 \cup \ldots \cup X_n)$, $\ldots$, $\mu(A_n) = \mu(X_n)$. The ranges $\{f(X_1), f(X_2), \ldots, f(X_n)\}$ are defined in ascending order as $f(X_1) \leqslant f(X_2) \leqslant \ldots \leqslant f(X_n)$, with the convention that $f(X_0) = 0$ and $\sum_{i=1}^{n} f(X_i) = 1$.

Example 6.2. From example 2.6, we can also compute special case of d-Choquet integral as shown below,

$$
\begin{aligned}
(O)\int_A f dg_\lambda &= \sum_{i=1}^{3}[f(X_i) - f(X_{i-1})]^2 g_\lambda(A_i) \\
&= [f(X_1) - f(X_0)]^2 \cdot g_\lambda(A_1) + [f(X_2) - f(X_1)]^2 \cdot g_\lambda(A_2) \\
&\quad + [f(X_3) - f(X_2)]^2 \cdot g_\lambda(A_3). \\
&= (0.4)^2 + (0.6 - 0.4)^2 \cdot 0.6275 + (0.8 - 0.6)^2 \cdot 0.3 = 0.1971.
\end{aligned}
$$

In the next example, we aim to demonstrate that the Sugeno and Choquet integrals do not adequately capture diversification, which is an important factor in MPT and as well decision-making. Instead, the d-Choquet integral addresses this limitation.

Example 6.3. In this example, we consider a portfolio consisting of securities such as bonds and stocks, with assigned haircut measures of 0.1 for bonds and 0.5 for stocks. The union of the stock and bond haircuts is equal to 1, indicating that they are mutually exclusive. Additionally, the expected returns for bonds and stocks are 0.05 and 0.1, respectively.
We examine three portfolios: A, B, and C. Portfolio A allocates 100% to bonds and 0% to stocks. Portfolio B allocates 75% to bonds and 25% to stocks, indicating a diversified allocation. Portfolio C allocates 0% to bonds and 100% to stocks.
Let us first summarise the above information into tables.

Expected Returns of the Financial Instruments

Securities	Expected returns
Bonds	0.05
Stocks	0.1

Risk of the Financial Instruments

Haircut Measure	Haircut value
m(Bonds)	0.1
m(Stocks)	0.5
m(Bonds $\cup$ stocks)	1

Portfolio Allocation

Portfolio	Bonds	Stock
Portfolio A	100% Bonds	0
Portfolio B	75% Bonds	25% Stocks
Portfolio C	0	100% Stocks

It is evident that portfolio B exhibits diversification by allocating a portion to both bonds and stocks. To assess whether the Sugeno and Choquet integrals capture this diversification, we will apply these aggregation functions to the portfolios.
In the table below are the results for the integrals.

Portfolio	Sugeno Integral	Choquet Integral	d-Choquet Integral	Expected return
Portfolio A	0.1	0.1	0.1	0.05
Portfolio B	0.25	0.3	0.0875	0.065
Portfolio C	0.5	0.5	0.5	0.1

We can compare the results of these aggregation functions with the diversified portfolio B and determine whether they adequately capture the benefits of diversification.
It is observed that the Sugeno integral indicates a small risk for portfolio A, followed by B, and then C. The Choquet integral also ranks the portfolios as A, B, and C in terms of risk. However,the d-Choquet integral highlights portfolio B as having the lowest risk, followed by portfolio A, and then portfolio C.
These findings align with the fact that portfolio B exhibits diversification through the allocation of both bonds and stocks. Therefore,the d-Choquet integral provides a more accurate representation of the risk associated with the portfolios, highlighting the benefits of diversification in portfolio B. This underscores the importance of considering diversification when assessing risk in investment portfolios.

6.4.1 Case study

In the case study, our focus is on the risk assessment and construction of efficient frontiers for a portfolio consisting of four financial instruments: X_1 representing bonds rated AAA, X_2 representing bonds rated BBB and better, X_3 representing stocks, and X_4 representing derivatives. For each instrument, haircut density values are provided to quantify the risk associated with them. These values reflect the reduction in the stated value of the instrument to account for potential losses.

To capture the inter-dependencies among the instruments, we compute their unions using the m_λ- measure as defined in Definition 2.6. These unions represent the combined risk factors across multiple instruments, considering their respective haircuts. By incorporating the unions, we gain a more comprehensive understanding of the overall risk profile of the investment portfolio.

6.4.2 Haircuts measures

Given the haircut densities $m_\lambda(X_1) = 0.1, m_\lambda(X_2) = 0.2, m_\lambda(X_3) = 0.5$ and $m_\lambda(X_4) = 0.95$, we apply gradient descent by the aid of a computer code presented in Section 6.2.4 to solve for the optimal value of λ and the corresponding m_λ measure values for various unions.

Input:

Inputting the haircut densities in the computer program, we can now compute λ and the corresponding m_λ measure values for various unions i.e,

```
m_lambda_measure([0.1,0.2,0.5,0.95]).
```

Output :

We obtain the value,

$\lambda = -0.971$.

Since we've λ , we can now compute m_λ measure values for various unions. The results are provided in the table below.

Haircut Measure	Haircut Value(Risk)
$m_\lambda(X_1)$	0.1
$m_\lambda(X_2)$	0.2
$m_\lambda(X_3)$	0.5
$m_\lambda(X_4)$	0.95
$m_\lambda(X_1 \cup X_2)$	0.2806
$m_\lambda(X_1 \cup X_3)$	0.5514
$m_\lambda(X_1 \cup X_4)$	0.9578
$m_\lambda(X_2 \cup X_3)$	0.6029
$m_\lambda(X_2 \cup X_4)$	0.9655
$m_\lambda(X_3 \cup X_4)$	0.9888
$m_\lambda(X_1 \cup X_2 \cup X_3)$	0.6444
$m_\lambda(X_1 \cup X_2 \cup X_4)$	0.9718
$m_\lambda(X_1 \cup X_3 \cup X_4)$	0.9928
$m_\lambda(X_2 \cup X_3 \cup X_4)$	0.9968
$m_\lambda(X_1 \cup X_2 \cup X_3 \cup X_4)$	1

The below table provides the expected returns for the financial instruments. These expected returns offer insights into the potential profitability or loss associated with each instrument.

Financial Instruments	Expected Returns
X_1	0.05
X_2	0.08
X_3	0.12
X_4	0.2

Subsequently, we assign weights to each asset within the portfolio, ensuring that the sum of all allocated weights amounts to 1, as the entire portfolio must be fully invested. For our analysis, we simulated a total of 256 portfolios using the provided four financial instruments. After obtaining the allocated weights, as shown in Table 1 (highlighted in green), we can proceed to compute the portfolio risk and portfolio expected return using the following formulas:

6.4.3 Portfolio Expected Return

The expected return of an investment or portfolio is the average return that an investor can anticipate over a given time period. It is calculated as the weighted sum of the individual asset returns or asset class returns in a portfolio. The formula for the expected return of a portfolio is:

$$E(P) = \sum_{i=1}^{n} w_i \cdot R_i$$

In the formular,
$E(P)$ represents the expected return of the portfolio.
n is the number of assets in the portfolio.
w_i is the weight (proportion) of asset i in the portfolio
R_i is the expected return of asset i

Now we compute the expected returns of each of the portfolios,

$$\begin{aligned}
E(P_1) &= f(X_1) \cdot 0.05 + f(X_2) \cdot 0.08 + f(X_3) \cdot 0.12 + f(X_4) \cdot 0.2 = 0.2 \\
E(P_2) &= f(X_1) \cdot 0.05 + f(X_2) \cdot 0.08 + f(X_3) \cdot 0.12 + f(X_4) \cdot 0.2 = 0.1 \cdot 0.12 + 0.9 \cdot 0.2 = 0.192 \\
E(P_3) &= f(X_1) \cdot 0.05 + f(X_2) \cdot 0.08 + f(X_3) \cdot 0.12 + f(X_4) \cdot 0.2 = 0.2 \cdot 0.12 + 0.8 \cdot 0.2 = 0.184 \\
E(P_4) &= f(X_1) \cdot 0.05 + f(X_2) \cdot 0.08 + f(X_3) \cdot 0.12 + f(X_4) \cdot 0.2 = 0.3 \cdot 0.12 + 0.7 \cdot 0.2 = 0.176 \\
E(P_5) &= f(X_1) \cdot 0.05 + f(X_2) \cdot 0.08 + f(X_3) \cdot 0.12 + f(X_4) \cdot 0.2 = 0.4 \cdot 0.12 + 0.6 \cdot 0.2 = 0.168 \\
&\vdots \\
E(P_{256}) &= f(X_1) \cdot 0.05 + f(X_2) \cdot 0.08 + f(X_3) \cdot 0.12 + f(X_4) \cdot 0.2 = 0.05.
\end{aligned}$$

6.4.4 Portfolio Risk

To determine the portfolio risk for each portfolio, we utilize the d-Choquet integral as described in Definition 6.2:

$$\begin{aligned}
(O)\int_{P_1} f dm_\lambda = {} & [f(X_1) - f(X_0)]^2 \cdot m_\lambda(X_1 \cup X_2 \cup X_3 \cup X_4) \\
& + [f(X_2) - f(X_1)]^2 \cdot m_\lambda(X_2 \cup X_3 \cup X_4) \\
& + [f(X_3) - f(X_2)]^2 \cdot m_\lambda(X_3 \cup X_4) \\
& + [f(X_4) - f(X_3)]^2 \cdot m_\lambda(X_4) = 0.95
\end{aligned}$$

$$\begin{aligned}
(O)\int_{P_2} f dm_\lambda = {} & [f(X_1) - f(X_0)]^2 \cdot m_\lambda(X_1 \cup X_2 \cup X_3 \cup X_4) \\
& + [f(X_2) - f(X_1)]^2 \cdot m_\lambda(X_2 \cup X_3 \cup X_4) \\
& + [f(X_3) - f(X_2)]^2 \cdot m_\lambda(X_3 \cup X_4) + [f(X_4) - f(X_3)]^2 \cdot m_\lambda(X_4) \\
& = (0.1)^2 \cdot 0.9888 + (0.9 - 0.1)^2 \cdot 0.95 \\
& = 0.617888
\end{aligned}$$

$$
\begin{aligned}
(O)\int_{P_3} f dm_\lambda &= [f(X_1)-f(X_0)]^2\cdot m_\lambda(X_1\cup X_2\cup X_3\cup X_4)\\
&+[f(X_2)-f(X_1)]^2\cdot m_\lambda(X_2\cup X_3\cup X_4)\\
&+[f(X_3)-f(X_2)]^2\cdot m_\lambda(X_3\cup X_4)+[f(X_4)-f(X_3)]^2\cdot m_\lambda(X_4)\\
&=(0.2)^2\cdot 0.9888+(0.8-0.2)^2\cdot 0.95\\
&=0.381552
\end{aligned}
$$

$$
\begin{aligned}
(O)\int_{P_4} f dm_\lambda &= [f(X_1)-f(X_0)]^2\cdot m_\lambda(X_1\cup X_2\cup X_3\cup X_4)\\
&+[f(X_2)-f(X_1)]^2\cdot m_\lambda(X_2\cup X_3\cup X_4)\\
&+[f(X_3)-f(X_2)]^2\cdot m_\lambda(X_3\cup X_4)+[f(X_4)-f(X_3)]^2\cdot m_\lambda(X_4)\\
&=(0.3)^2\cdot 0.9888+(0.7-0.3)^2\cdot 0.95\\
&=0.240992
\end{aligned}
$$

$$
\begin{aligned}
(O)\int_{P_5} f dm_\lambda &= [f(X_1)-f(X_0)]^2\cdot m_\lambda(X_1\cup X_2\cup X_3\cup X_4)\\
&+[f(X_2)-f(X_1)]^2\cdot m_\lambda(X_2\cup X_3\cup X_4)\\
&+[f(X_3)-f(X_2)]^2\cdot m_\lambda(X_3\cup X_4)+[f(X_4)-f(X_3)]^2\cdot m_\lambda(X_4)\\
&=(0.4)^2\cdot 0.9888+(0.6-0.4)^2\cdot 0.95\\
&=0.196208
\end{aligned}
$$

$\vdots$

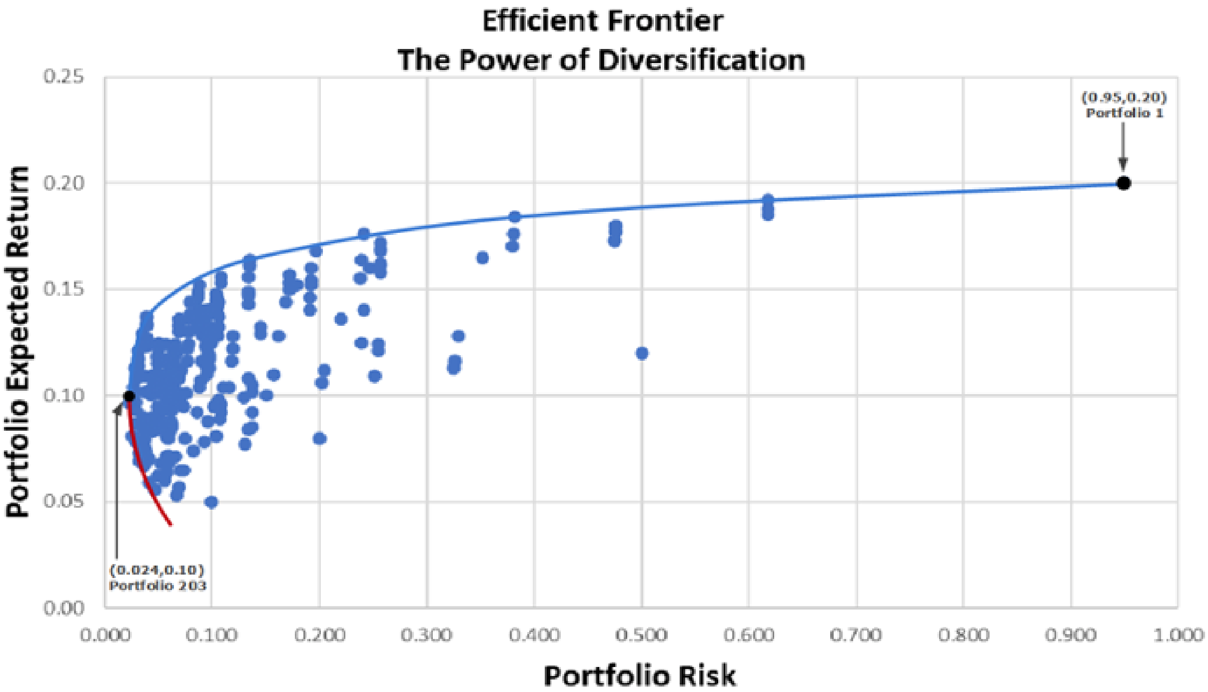
$$
\begin{aligned}
(O)\int_{P_{256}} f dm_\lambda &= [f(X_4)-f(X_3)]^2\cdot m_\lambda(X_1\cup X_2\cup X_3\cup X_4)\\
&+[f(X_3)-f(X_2)]^2\cdot m_\lambda(X_1\cup X_2\cup X_3)\\
&+[f(X_2)-f(X_1)]^2\cdot m_\lambda(X_2\cup X_1)+[f(X_2)-f(X_1)]^2\cdot m_\lambda(X_1)\\
&=1\cdot 0.1\\
&=0.1.
\end{aligned}
$$

The table below summarizes the results for the portfolio risk and expected return values for each portfolio.

Table 1: Portfolio risk and expected return

Portfolio	$f(X_1)$	$f(X_2)$	$f(X_3)$	$f(X_4)$	Portfolio Expected Return	Portfolio Risk
Portfolio 1	0	0	0	1	0.2	0.95
Portfolio 2	0	0	0.1	0.9	0.192	0.617888
Portfolio 3	0	0	0.2	0.8	0.184	0.381552
Portfolio 4	0	0	0.3	0.7	0.176	0.240992
Portfolio 5	0	0	0.4	0.6	0.168	0.196208
Portfolio 6	0	0	0.5	0.5	0.16	0.2472
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Portfolio 255	0.9	0.1	0	0	0.053	0.066806
Portfolio 256	1	0	0	0	0.05	0.1

The analysis of the portfolio data presented in the table above has provided valuable insights into the risk-return characteristics of various investment options. By factoring in the non-additive risk measures in portfolio analysis, we gain a deeper understanding of the risk dynamics within the portfolio, facilitating more informed and robust decision-making. Below is the graph to show the efficient frontier with optimal portfolios.



All portfolios below the efficient frontier curve are sub-optimal, as they do not offer the highest expected return for a given level of risk. Only those portfolios on the curve are optimal. Portfolio 203 has the smallest risk and at the same level of risk with other portfolios,it maximizes the returns. This means that it is the portfolio with the highest Sharpe ratio, which is a measure of the return per unit of risk. Portfolio 1 has the highest risk and highest returns.

The selection of portfolios ultimately hinges upon individual investors' risk preferences and investment objectives. Risk-averse individuals are more inclined to invest in portfolios like Portfolio 203 or those in close proximity on the efficient frontier, which offer optimal returns at reduced risk levels.On the other hand, risk seekers may opt for portfolios like Portfolio 1 or those in close proximity drawn to the potential for higher returns despite the accompanying elevated risk.

Ultimately, Modern Portfolio Theory highlights the importance of constructing portfolios that align with individual risk tolerances, financial goals, and time horizons. Whether one aims for stability or embraces greater risk, understanding the dynamics of the efficient frontier empowers investors to make informed and strategic investment decisions tailored to their unique preferences and objectives.

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