NEIGHBORHOOD AND TOPOLOGICAL MODELS OF CLASSICAL AND INTUITIONISTIC MODAL LOGICS

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Abstract. This is a brief summary of a doctoral dissertation. The author's purpose is to present both his aims and achievements in the field of mathematical logic and general topology.

1. Introduction

In this summary we would like to emphasize the most important results of our doctoral thesis entitled *Modele otoczeniowe i topologiczne dla klasycznych i intuicjonistycznych logik modalnych* (eng. *Neighborhood and topological models of classical and intuitionistic modal logics*). This thesis has been prepared by the author during his doctoral studies at the University of Silesia (between 2014 and 2020). The author's branch of science is *mathematics*. Actually, he is interested in *mathematical logic*, especially in the semantics of various non-classical logics. His supervisor is Mr Tomasz Połacik, Ph.D. Assoc. Prof.

The author has already published three scientific papers (excluding pre-prints on `arxiv.org`). Moreover, he participated in a dozen of conferences and workshops. He holds a Master of Science degree in Mathematics, graduated from the University of Wroclaw in 2011 (his Master Thesis was about analysis in the field of p-adic numbers).

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2. Formulation of the topic

First of all, we may speak about syntax. From this point of view any logic can be considered as as the set of axioms and rules. Here we are interested in formal proofs and deduction systems. Second, we can also think about semantics, namely, about some models in which it is possible to define the notions of truth and falsity.

As for the logical calculi, we are working with *propositional logics*. Thus, we are not so much interested in quantifiers. Our logics are non-classical. Of course, there are many kinds of non-classical logic and many reasons for which certain system can be considered as non-classical. In our case, there are two main ways which are notoriously combined. On the one hand, we are interested in intuitionistic, super-intuitionistic and subintuitionistic systems. This means that we narrow down the set of axioms and rules of classical logic. On the other hand, we use modal operators to define and analyse the ideas of necessity and possibility. As a result, we often obtain *classical* and *intuitionistic modal logics*.

Our semantic models are mostly neighborhood, topological and relational. These three approaches are also combined. For this reason, we may speak about bi-relational and relational-neighborhood structures. Moreover, we go beyond the standard notion of topology in order to study its various generalizations.

Finally, our aim is to investigate several non-classical calculi using all the tools mentioned above. We are interested in the issues of completeness (axiomatization), finite model property, bisimulation and decidability. Moreover, we analyse some purely topological properties of the structures in question. The philosophical aspect is also important.

3. Justification of the study

Let us start from the importance of non-classical logics. First of all, they have many applications: from philosophy to physics. They are widely used to speak about all these concepts which are far beyond the scope of classical logic: like necessity and possibility, probability, uncertainty, ambiguity etc. Also, they are closely related to some algebraic and topological structures known from the other branches of mathematics (for example, there is a strict correspondence between intuitionistic logic and Heyting algebras). Hence, it is clear that the very idea of studying non-classical logics is sensible. Then what about our particular semantic tools?

Topological notions are very useful in formal logic, without any doubt. They form bridge between possible-world semantics (which can be considered as somewhat abstract) and well-known mathematical objects (like real line, real plane or Cantor set). Topology allows us to discuss various properties of possible-world frames (depending on axioms of separation or on the notions of density, compactness etc.). Moreover, sometimes these properties can be characterized by means of specific formulas.

On the other hand, topology is rather strong notion. For example, topological semantics for modal logics leads us to systems not weaker than **S4** . They are equivalent with the so-called **S4** *neighborhood frames* (see [6]). However, neighborhoods are most frequently used with non-normal logics, sometimes very weak. The problem is that in topology *neighborhood* is very rigorous notion, while in possible-worlds semantics it is just an arbitrary (maybe even empty) collection of worlds which are *assigned* to the given world w.

For this reason, it is difficult to speak about topological semantics for logics weaker than **S4** , not to mention non-normal systems. Hence, we use some generalized concepts like *generalized* topological spaces of Császár, infra-topological spaces, pseudo-topologies etc. They can be considered as a substitute of topological frames in the context of weak modal logics. Moreover, we use subspace topologies when speaking about intuitionistic modal logic.

We dare to think that this approach, albeit not revolutionary or groundbreaking, opens some new ways and gives us a new look at the question of modal logics and their semantics. There is also a clarification of some basic notions.

4. Methodology

We are interested primarily in semantics, i.e. in frames, models and the meaning or interpretation of modal operators. We often start from some well-known structure in order to find its new version which will be useful in the new context, e.g. in the context of non-normal calculi. Moreover, we try to state some natural questions. For example: recently, there has been established (see [4]) neighborhood semantics for intuitionistic propositional logic. In fact, it has been based on the notion of *minimal* neighborhood, while the *maximal* one has been simply identified with the whole universe (namely, with the set of all worlds). Thus, the natural question appears: is it reasonable to assume that our world has not only minimal but also maximal (different than the whole universe) neighborhood? What would it mean for the formula to be satisfied in this maximal neighborhood? Is there a place for modal operator, say, the one of necessity?

Another example: if we know that topological spaces correspond to **S4** logic, then maybe it would be cognitively valuable to check which results can be reproduced in the weaker environment of generalized topologies and non-normal logics?

These and similar considerations led us to the formulation of some theorems about frames and models which are investigated in this thesis.

5. Basic notions

Here we would like to list some basic notions which are frequently used in our dissertation. We are concentrated on those which are taken from the general vault of mathematical logic and topology (excluding our own, specific and new terms). Hence, the reader should be at least basically acquainted with the following concepts:

- Non-classical propositional logics.
- Intuitionistic modal logic.
- Weak modal logics: non-normal, non-regular and non-monotonic.
- Subintuitionistic logics.
- Hilbert-style formulation of propositional logics.
- Possible worlds semantics in its many versions: relational, bi-relational, neighborhood.
- Finite model property, filtration, decidability.
- Soundness and completeness of logical calculus; Henkin method, maximal and prime theories.
- Bounded morphism, bisimulation.
- The very idea of topological space; the awareness of the fact that it is possible to formulate generalizations of this basic notion.
- Fuzzy sets and other methods of modelling uncertainty.
- Continuity, sequence, net, convergence.

Moreover, there are some modal axioms (and general rules of inference) which should be known:

- $M : \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
- $C : \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
- $T : \Box\varphi \rightarrow \varphi$
- $H : \Box\varphi \vee \Box\psi \rightarrow \Box(\varphi \vee \psi)$.
- $D : \Box \rightarrow \neg\Box\neg\varphi$

- $K : \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $4 : \Box\varphi \to \Box\Box\varphi$
- $N : \Box\top$
- $RE : \varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi$
- $RN : \varphi \vdash \Box\varphi$
- $RM : \varphi \to \psi \vdash \Box\varphi \to \Box\psi$
- $MP : \varphi, \varphi \to \psi \vdash \psi$

It is worth to know which notions and concept are *beyond* the scope of our thesis. We do not study multi-valued and relevance logics. Paraconsistent systems are mentioned only briefly. We do not deal (at least not directly) with algebraic semantics. As for the syntax and proof theory, we are interested only in Hilbert-style formulations, hence we do not discuss natural deduction or sequent calculi. As we have already said, we stay on the propositional (sentential) level which means that we do not investigate first- and second-order systems.

6. Overview of chapters

Our thesis consists of introduction and six *chapters* (each chapter consists of *sections* and *subsections*).

In the **first** chapter, we modify neighborhood semantics for intuitionistic logic as it was introduced by Moniri and Maleki in [4]. We assume that each possible world has minimal and maximal neighborhood. The former simulates intuitionistic reachability of other worlds, the latter refers to the modal aspect.

Our basic structure is a neighborhood modal frame (**in1**-frame) defined as it follows:

Definition 6.1. **in1**-frame is an ordered pair $\langle W, \mathcal{N} \rangle$ where:

1. W is a non-empty set (of worlds, states or points)
2. $\mathcal{N}$ is a function from W into $P(P(W))$ such that:
   (a) $w \in \bigcap \mathcal{N}_w$
   (b) $\bigcap \mathcal{N}_w \in \mathcal{N}_w$
   (c) $u \in \bigcap \mathcal{N}_w \Rightarrow \bigcap \mathcal{N}_u \subseteq \bigcap \mathcal{N}_w$ (*$\to$-condition*)
   (d) $X \subseteq \bigcup \mathcal{N}_w$ and $\bigcap \mathcal{N}_w \subseteq X \Rightarrow X \in \mathcal{N}_w$ (*relativized superset axiom*)
   (e) $u \in \bigcap \mathcal{N}_w \Rightarrow \bigcup \mathcal{N}_u \subseteq \bigcup \mathcal{N}_w$ (*$\Box$-condition*).

As for the valuation of propositional variables, we assume that it is monotone (i.e. it hold in the whole minimal neighborhood of a given world w, not only in the world itself). Forcing of complex formulas is defined inductively, wherein:

(1) $w \Vdash \varphi \to \psi \Leftrightarrow \bigcap \mathcal{N}_w \subseteq \{v \in W; v \nVdash \varphi \text{ or } v \Vdash \psi\}$.
(2) $w \Vdash \Box\varphi \Leftrightarrow \bigcup \mathcal{N}_w \subseteq \{v \in W; v \Vdash \varphi\}$.

We show that intuitionistic modal logic $\mathbf{iKT}_\Box$ is sound and complete with respect to this semantics. We prove it directly (using canonical model) and indirectly (showing that our structures are pointwise equivalent with bi-relational frames introduced in [1]). Then we use filtration to prove finite model property and decidability. Later we consider the notions of behavioral equivalence, bounded morphism, bisimulation and n-bisimulation (the last one is quite complex). Moreover, we introduce some new operators: additional implication $\rightsquigarrow$, possibility operator $\Diamond$ and public announcement modality. Finally, we show that our initial structures are compatible with neighborhood models for some classical logics equipped with two necessity operators (the first one is taken from **K** and the second one from **S4**).

In the **second** chapter we present three examples of topological semantics for $\mathbf{iKT}_\Box$. We show that it is possible to treat neighborhood models, introduced earlier, as topological (or *multi-topological*). From the neighborhood point of view, our method is based on differences between properties of minimal and maximal neighborhoods. Also we propose transformation of these multi-topological spaces into the neighborhood structures. Our first intuition was that neighborhood systems assigned to the particular worlds behave like distinct topological spaces in a kind of "meta-universe". We show initial conclusions of this observation. However, in some cases it is better to assume that all these systems are in fact subspaces of one topological space. Hence, we can use the notion of *induced topology*. Moreover, we introduce the idea of topological space with *distinguished* sets (such distinguished set contains w and plays the role of its minimal neighborhood).

In the **third** chapter we analyse logic of false belief in the intuitionistic setting. This logic, studied in its classical version by Steinsvold, Fan ([3]), Gilbert and Venturi, describes the following situation: a formula φ is not satisfied in a given world, but we still believe in it (or we think that it should be accepted). Another interpretations are also possible: e.g. that we do not accept φ but it is imposed on us by a kind of council or advisory board. From the mathematical point of view, the idea is expressed by an adequate form of modal operator W which is interpreted in relational frames with neighborhoods:

$w \Vdash \mathsf{W}\varphi \Leftrightarrow w \Vdash \neg\varphi$ and $V(\varphi) \in \mathcal{N}_w$.

We discuss monotonicity of forcing, soundness, completeness and several other issues. We present also some simple systems in which confirmation of previously accepted formula is modelled.

The next chapter, namely, the **fourth** one, deals with various generalizations of the notion of topology. We start with a survey of already discovered structures. Among them are: Császár's topologies (i.e. families of subsets closed under arbitrary unions, see [2]), infra-topologies, minimal structures, weak structures, generalized weak structures and other ones. Moreover, we discuss intuitionistic, rough and binary topologies. Later we concentrate on Császár's spaces. What is important, is the fact that the whole universe of generalized topological space may not be open. Hence, some points may be beyond any open set. We assume that such points are associated with certain open neighbourhoods by means of a special function $\mathcal{F}$. This leads to the formulation of the notion of $\mathbf{GT}\mathcal{F}$-structure.

Definition 6.2. We define $\mathbf{GT}\mathcal{F}$-structure as a triple $M_\mu = \langle W, \mu, \mathcal{F} \rangle$ such that μ is a generalized topology on W and $\mathcal{F}$ is a function from W into $P(P(\bigcup \mu)$ such that:

- If $w \in \bigcup \mu$, then $[X \in \mathcal{F}_w \Leftrightarrow X \in \mu$ and $w \in X]$ [$\mathcal{F}_w$ is a shortcut for $\mathcal{F}(w)$].
- If $w \in W \setminus \bigcup \mu$, then $[X \in \mathcal{F}_w \Rightarrow X \in \mu]$.

However, function $\mathcal{F}$ is very general and vague. For this reason, we consider also more precise function $\mathbf{f}$: the idea is that each world from $W \setminus \bigcup \mu$ inherits open neighborhoods from some "sister" world in $\bigcup \mu$.

We use all these notions in logical context. We show that our *generalized topological models* (for non-normal modal logics) are compatible with certain subclass of neighborhood models. We compare our results with those of Soldano (who investigated *extensional abstractions*) as well as Järvinen, Kondo and Kortelainen (who spoke about *interior systems*). Moreover, we discuss the notion of bisimulation in several contexts. Then we consider the notion of *impossible* worlds in our frames. Finally, we use our semantics as a model of certain subintuitionistic logic (without modal operators).

The **fifth** chapter is about infra-topologies. We adhere to the definition of *infra-topological* space as it was introduced by Al-Odhari in [5]. Namely, we speak about families of subsets which contain $\emptyset$ and the whole universe X, being at the same time closed under finite intersections (but not necessarily under arbitrary or even finite unions). This slight modification allows us to distinguish between new classes of subsets (infra-open, ps-infra-open and i-genuine). Analogous notions are discussed in the language of closures. The class of minimal infra-open sets is studied too, as well as the idea of generalized infra-spaces. Finally, we obtain characterization of infra-spaces in terms of modal logic, using some of the notions introduced above.

In the **sixth** chapter we study various topological properties of the $\mathbf{GT}\mathcal{F}$-structures. We introduce the notions of $\mathcal{F}$-interior and $\mathcal{F}$-closure and we discuss issues of convergence in this new setting. We are interested primarily in generalized nets. While sequences have natural domain and nets have directed domain, generalized nets (*gnets*) have pre-ordered domains.

7. Further research

As for the further studies, we are interested mostly in:

- Generalized topological semantics for subintuitionistic logics (with and without modalities).
- Intuitionistic version of the logic of uknown truths. Moreover, it would be good to connect logics of false belief and unknown truths with some paraconsistent tools. We think about operators of underterminancy (N) and ambiguity (M), invented and investigated by Żabski in [7].
- Some applications of double (flou) sets in the theory of negotiations. Our initial results in this area have been presented in the fourth chapter of our thesis. We combine standard definition of double set with operations which are typical for intuitionistic sets. As a result, we obtain a structure of discussion between several participants who propose their "necessary" and "allowable" requirements or propositions.
- Further topological properties of infra-topologies, $\mathbf{GT}\mathcal{F}$- and $\mathbf{GTf}$-structures as well as other weak spaces. We are interested in density, nowhere density, convergence and continuity.

8. Publications

The author has already published three official papers:

(1) T. Witczak, *Generalized topologies with associating function and logical applications*, Acta et Commentationes Universitatis Tartuensis de Mathematica, Vol. 24, Number 2, December 2020.
(2) T. Witczak, *Propositional logic with probability operators (based on general ideas of weak modal calculus)*, in: *Reasoning: games, cognition, logic*, red. M. Urbański, T. Skura, P. Łupkowski, w serii *Studies in Logic*, vol. 83, College Publications 2020.
(3) T. Witczak, *Topological and multi-topological frames in the context of intuitionistic modal logic*, Bulletin of the Section of Logic, vol. 48, no. 3 (2019), `https://czasopisma.uni.lodz.pl/bulletin/article/view/6205/5833`.

Besides, he has several pre-prints on arxiv.org, of which the most important are:

(1) T. Witczak, *A note on the intuitionistic logic of false belief*, `https://arxiv.org/pdf/2012.08309.pdf`.
(2) T. Witczak, *Infra-topologies revisited: logic and clarification of basic notions*, `https://arxiv.org/pdf/2012.03558.pdf`.
(3) T. Witczak, *Negotiation sets: a general framework*, `https://arxiv.org/pdf/2102.04982.pdf`.

Those papers have been sent to the journals and are currently under review.

Moreover, the author himself reviewed two papers for the following journals: *The Mathematics Student* (Indian Math. Society, `http://www.indianmathsociety.org.in/ms.htm`) and *Communications of the Korean Mathematical Society* (`https://ckms.kms.or.kr/`). Now he is working on the review for *Journal of Logic and Computation*.

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